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The chiral anomaly from M theory

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Abstract

We argue that the chiral anomaly of $\mathcal{N} = 1$ super Yang-Mills theory admits a dual description as spontaneous symmetry breaking in M theory on G_2 holonomy manifolds. We identify an angle of the G_2 background dual to the anomalous $U(1)_R$ current in field theory. This angle is not an isometry of the metric and we therefore develop a theory of “massive isometry” to describe fluctuations about such angles. Another example of a massive isometry occurs in the Atiyah-Hitchin metric.

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1 Background and motivation

Dualities between nonconformal $\mathcal{N} = 1$ gauge theories and gravity theories on supersymmetric backgrounds continue to provide important insight into the dynamics of strongly coupled gauge theories with similar properties to Quantum Chromodynamics.

In this paper we explore these dualities further. Specifically, we work towards placing dualities involving G_2 holonomy M theory backgrounds on a comparable footing to those introduced by Klebanov-Strassler [1] and Maldacena-Núñez [2]. An important property of G_2 holonomy solutions is that classically they are purely gravitational, with no fluxes.

We will match a dynamical phenomenon on the gravity and gauge theory sides of the duality. Namely, we argue that the chiral anomaly of $\mathcal{N} = 1$ super Yang-Mills (SYM) theory is dual to spontaneous symmetry breaking in the G_2 background. In order to make this matching we introduce the concept of a “massive isometry” of a gravity background. We develop this idea further in the later parts of the paper.

Many questions require further investigation, we mention some of these in the final discussion.

1.1 $\mathcal{N} = 1$ backgrounds and the G_2 holonomy challenge

The original Anti-de Sitter / Conformal Field Theory (AdS/CFT) correspondence [3, 4, 5] showed that the idea of a duality between gauge theory and gravity could be realised by critical string theory. Given this, an important question is whether string theory can provide dual descriptions to strongly coupled nonconformal theories with reduced supersymmetry.

Initial progress in this direction came through understanding deformations of $\mathcal{N} = 4$ field theory [6, 7], see [8] for a review. Deformations involve either nonzero vacuum expectation values in the field theory or the addition of relevant or marginal operators to the action. The resulting five dimensional gravity backgrounds, after dimensional reduction on any compact internal space, are asymptotic to AdS spacetime, corresponding to the deformations becoming negligible in the ultraviolet (UV) of the field theory.

These cases were then generalised by the ‘geometrical engineering’ within string theory of gravity duals to field theories without a UV conformal fixed point. Two completely regular backgrounds are known, which provide duals to nonconformal $\mathcal{N} = 1$ field theories. One is the Klebanov-Strassler background [1], constructed using fractional branes placed at a conifold singularity. The other is the Maldacena-Núñez background [2], constructed as a supersymmetric configuration of D5-branes wrapped on an S^2 inside a Calabi-Yau manifold.

A recent clear review of nonconformal dualities with $\mathcal{N} = 1$ and $\mathcal{N} = 2$ supersymmetry is [9]. From a ten dimensional string theory perspective, the duality we investigate here is conceptually similar to a IIA version of the Maldacena-Núñez background, where now we have D6-branes wrapping an S^3 . However, there does not appear to be a direct connection via string theory dualities between the backgrounds we consider and the Maldacena-Núñez background.

It is important to check the genericity of predictions from gravity duals by studying new examples with different properties. As we just mentioned, an alternative way to geometrically engineer $\mathcal{N} = 1$ SYM theory is to wrap D6-branes on an S^3 inside a Calabi-Yau manifold [10]. An advantage of this setup, compared with the IIB setup of wrapped D5-branes, is that the resulting configurations have a very simple lift to M theory: they are described by purely gravitational backgrounds with G_2 holonomy [11]. The M theory perspective proved useful in clarifying the string theory duality of [10] in terms of a topology change in M theory. This process was further systematised at the quantum level by Atiyah and Witten [12] who were able to control quantum corrections to show that the topology change was smooth. The connection between $\mathcal{N} = 1$ SYM theory and G_2 holonomy manifolds was further developed by studying the structure of ADE singularities in such manifolds [13, 14].

In parallel with the development of physical intuition associated with G_2 holonomy, there has been important progress recently in constructing new explicit cohomogeneity one G_2 metrics. The original construction of G_2 metrics in [15, 16] was generalised to cases where the dimensionally reduced IIA dilaton remained finite asymptotically [17]. This was later further generalised to cases which shared the improved asymptotic behaviour and further had no orbifold singularity at the origin [18, 19, 20], corresponding perhaps to backgrounds giving a good infra-red (IR) description of gauge theory physics.

Despite such major progress regarding G_2 holonomy, the understanding of a relationship with $\mathcal{N} = 1$ gauge theory and the construction of explicit metrics have not yet coalesced into the development of a duality along the lines of the Klebanov-Strassler or Maldacena-Núñez backgrounds.

Various topological quantities have been matched between the gravity and field theory descriptions. Thus wrapped membranes in the IR geometry describe gauge theory confining strings [14, 21] and wrapped fivebranes on three-cycles describe gauge theory domain walls [14, 22]. However, the G_2 metrics themselves, which will be needed to match dynamical phenomena, are not used in these calculations. Some properties of G_2 metrics were used

by Atiyah and Witten [12] in calculating the space of asymptotically G_2 vacua, although the metrics used were old asymptotically conical cases [15, 16] that do not admit a IIA reduction with everywhere finite dilaton. In contrast, the physics we describe will depend crucially on the leading and subleading asymptotic form of the recently constructed metrics.

As far as dynamics are concerned, a recent study of rotating membranes in explicit G_2 backgrounds [23] reproduced a logarithmic relationship between spin and energy that is characteristic of twist two operators in gauge theories. However, it was not clear how to concretely match this tantalising result with gauge theory quantities.

We argue here that the chiral anomaly in field theory admits a dual description as spontaneous symmetry breaking of a gauge symmetry in M theory. This enables us to explicitly identify a circle in the recently constructed G_2 metrics that is dual to the chiral symmetry of the field theory.

1.2 The chiral anomaly in gauge theory

Let us briefly recall the origin of the chiral anomaly in $\mathcal{N} = 1$ Yang-Mills theory. We follow the succinct and fully nonperturbative exposition of [24].

Pure $\mathcal{N} = 1$ Yang-Mills theory in four dimensions contains a massless spin-half gluino field, Ψ , that transforms in the adjoint representation of the gauge group. The Lagrangian

$$\frac{i}{g^2} \bar{\Psi} \not{D}_{\text{adj}} \Psi, \quad (1)$$

is classically invariant under the $U(1)_R$ chiral symmetry

$$\Psi \rightarrow e^{i\alpha\gamma_5} \Psi, \quad (2)$$

where as usual γ_5 has eigenvalues ± 1 acting on positive and negative chirality spinors respectively, and $0 \leq \alpha < 2\pi$ parametrises the $U(1)$ group. The notation $U(1)_R$ means that the chiral symmetry in this case is the R symmetry of the $\mathcal{N} = 1$ supersymmetry algebra.

The quantum mechanical anomaly may be seen from the nontrivial change in the path integral measure under the transformation (2)

$$D\Psi \rightarrow \frac{D\Psi}{\det e^{i\alpha\gamma_5}} = D\Psi e^{-i\alpha \text{tr} \gamma_5} = D\Psi e^{-i\alpha \text{Ind} \not{D}_{\text{adj}}}. \quad (3)$$

The steps used here are as follows. The first step is simply the Jacobian of the chiral transformation (2). The second uses the standard relation between the determinant and trace of operators. Note that the trace of γ_5 is not over spinor matrix indices but rather over state space. The final step uses the fact that the Dirac index is defined as the difference

between the number of positive and negative chirality massless modes, and that massive modes always come in pairs of positive and negative chirality and so do not contribute to the trace of γ_5 .

If the gauge group is $SU(N)$, the index of the Dirac operator in the adjoint representation is an integral multiple of $2N$. Therefore, the surviving symmetry is \mathbb{Z}_{2N} , corresponding to values $\alpha = (2\pi n)/(2N)$, with $n \in 0 \dots 2N$.

We have included this short derivation to emphasise the fact that a nonperturbative treatment is possible. One could obtain the same results from the standard perturbative one-loop triangle diagrams. The protection of the chiral anomaly from nonperturbative corrections is ultimately why the phenomenon is amenable to study in dualities.

1.3 Dual descriptions of the chiral anomaly

A dual description of the anomaly in the chiral $SU(4)_R$ symmetry in $\mathcal{N} = 4$ super Yang-Mills, with external gauged $SU(4)$ sources, was amongst the first achievements of the AdS/CFT duality [5]. The principal steps in the matching are as follows. Firstly, that the gravity background has a gauge field excitation A_μ that couples to the chiral symmetry current J^μ of the dual theory. Secondly, the anomaly in the field theory, $\partial \cdot J \neq 0$, translates into a breaking of gauge invariance in the gravity solution.

In the $\mathcal{N} = 4$ case, the breaking of gauge invariance in the gravity solution occurs due to boundary terms that arise because of a Chern-Simons term in the supergravity action [5]. The only other known mechanism of breaking gauge invariance in a physically sensible way is through spontaneous symmetry breaking.

Spontaneous symmetry breaking was first related to the chiral symmetry in the study of deformations of the AdS/CFT correspondence [25, 26]. The gravity duals are described by a five dimensional gauged supergravity in an asymptotically AdS background. In the case of flow along the Coulomb branch of the $\mathcal{N} = 4$ theory, one sees that the R-symmetry $SO(6) = SU(4)$ is broken in the interior of the spacetime to $SO(p)$, $p \leq 6$, by some of the $SO(6)$ gauge fields acquiring a mass [26]. In these cases, the breaking of chiral symmetry in the field theory is not due to anomaly. Instead, the symmetry is explicitly broken by vacuum expectation values or by deformations.

The use of spontaneous symmetry breaking as dual to breaking of chiral symmetry was extended to anomalous, as opposed to explicit, breaking in [27, 28]. The duality in question involved the Klebanov-Strassler $\mathcal{N} = 1$ solution. The key idea, reviewed in the next subsection, works as follows. The asymptotic, UV, metric has a $U(1)$ isometry. When this

direction is perturbed, the Lagrangian for the fluctuation is described by the usual gauge-kinetic term, F^2 , for a vector field. However, in order for the perturbation to be consistent, the background Ramond-Ramond fields must also be perturbed, as well as the metric. These contribute an extra mass term to the Lagrangian for the fluctuation. The resulting Lagrangian for a massive vector field may be understood as the result of spontaneous symmetry breaking. In the spontaneous breaking, the gauge field ‘eats’ a scalar field to become massive. The scalar field itself does not acquire a nonzero vacuum expectation value in order to break the symmetry. Strictly speaking the scalar is thus not a Higgs field, but rather a Stückelberg scalar field. Reproducing this phenomenon for the G_2 holonomy duality, where there are no p-form field strengths, will be the main objective of this work.

Another aspect of the chiral anomaly that is visible in the Klebanov-Strassler and Maldacena-Núñez backgrounds is the breaking of $U(1)$ to \mathbb{Z}_{2N} due to field theory anomaly in the UV, and then the breaking of \mathbb{Z}_{2N} to \mathbb{Z}_2 in the IR [2, 27, 29].

1.4 Spontaneous symmetry breaking in the Maldacena-Núñez background

Rather than review literally the argument of [27, 28] for the Klebanov-Strassler background, we show that it is trivially adapted to the Maldacena-Núñez (MN) background [2]. In fact it works more cleanly, because there is no self-dual five form. The result is interesting in its own right because it provides an understanding of the chiral symmetry breaking in the MN solution as spontaneous symmetry breaking due to non-invariance of the Ramond-Ramond (RR) two-form potential under the chiral isometry.

The MN solution considers D5-branes wrapping the topologically non trivial two-cycle, S^2 , within the resolved conifold, a Calabi-Yau three-fold. In the spirit of the gravity/gauge theory correspondence, the field theory on the D-brane worldvolume in a certain regime, is dual to the supergravity solution generated by the gravitating D-branes. For the MN configuration, the field theory is $\mathcal{N} = 1$ SYM theory. When the backreaction of the branes is taken into account, the two-cycle, S^2 , becomes topologically trivial and a three-cycle, S^3 , blows up. Thus the dual geometry is the deformed conifold with non-vanishing fluxes.

Since the anomalous breaking of chiral symmetry is a UV effect in field theory, consider the geometry at large r , in the Einstein frame and with $\alpha' = 1$:

$$ds_{10}^2 = e^{\Phi/2} \left[dx_{1,3}^2 + N \left(dr^2 + r[d\theta^2 + \sin^2 \theta d\varphi^2] + \right. \right. \\ \left. \left. + \frac{1}{4}[d\alpha^2 + \sin^2 \alpha d\beta^2 + (d\psi + \cos \alpha d\beta - \cos \theta d\varphi)^2] \right) \right], \quad (4)$$

where the dilaton at large r is,

$$e^{2\Phi} = e^{2\Phi_0} \frac{e^{2r}}{4\sqrt{r}}. \quad (5)$$

There is the following flux

$$F_3 = -\frac{1}{4}N \left[\omega_2 \wedge d\psi - \cos\theta \sin\alpha d\alpha \wedge d\beta \wedge d\varphi + \sin\theta \cos\alpha d\theta \wedge d\varphi \wedge d\beta \right], \quad (6)$$

where ω_2 is the volume form on a two-cycle

$$\omega_2 = \sin\alpha d\alpha \wedge d\beta - \sin\theta d\theta \wedge d\varphi. \quad (7)$$

The RR two-form potential is

$$C_2 = \frac{1}{4}N \left[\psi \omega_2 + \cos\theta \cos\alpha d\varphi \wedge d\beta \right]. \quad (8)$$

The metric (4) clearly exhibits the isometry

$$\psi \rightarrow \psi + \epsilon. \quad (9)$$

In the same spirit as [27] we propose this isometry as the dual of the R-symmetry in the UV limit of $\mathcal{N} = 1$ SYM and call it the “chiral isometry” from now on. This is a symmetry of the RR field strength (6), although not of the potential (8). In section 5.3 below, this fact will result in the breaking $U(1) \rightarrow \mathbb{Z}_{2N}$.

We will consider a fluctuation about the chiral isometry. The presence of the field strength will result in the usual vector perturbation obtaining a mass through spontaneous breaking. This is crucial for the gauge theory/gravity correspondence because breaking of a global symmetry in the field theory by any means (explicit, anomalous or spontaneous) should be sought as a spontaneous symmetry breaking of the dual gauge symmetry on the gravity side.

The perturbation of the metric (4) and potential (8) is

$$d\psi \rightarrow d\psi + A_a(x, r) dx^a, \quad \psi \rightarrow \psi + \lambda(x, r), \quad (10)$$

where the index a runs over the five coordinates t, x, y, z, r . Introducing this perturbation is just a gauging of the chiral isometry, because now the coordinate transformation $\psi \rightarrow \psi + \epsilon(x, r)$ corresponds to the $U(1)$ gauge transformations

$$A(x, r) \rightarrow A(x, r) + d\epsilon(x, r), \quad \lambda(x, r) \rightarrow \lambda(x, r) + \epsilon(x, r). \quad (11)$$

Note that we needed to perturb the RR potential in order to gauge the isometry.

As usual, the Ricci scalar of the perturbed metric is given the original value plus a gauge-kinetic term

$$R(A) = R(A=0) - \frac{1}{16} N e^{\Phi/2} |F_2|^2, \quad (12)$$

where $F_2 = dA$. The three-form field strength becomes

$$F_3(\lambda) = F_3(\lambda=0) - \frac{1}{4} N (\omega_2 \wedge d\lambda). \quad (13)$$

At this point, one should introduce the Stückelberg-Pauli vector field $W = A - d\lambda$ which is gauge invariant. In terms of the gauge invariant quantity W we obtain

$$|F_3(\lambda)|^2 = |F_3(0)|^2 + \frac{3(1+16r^2)}{8r^2 N} e^{-\Phi} N W_a W^a. \quad (14)$$

Note that the calculation is rather remarkable. All the components of the gauge field A come from the metric, while the components of λ come from the three-form. They rearrange themselves exactly to produce the mass term for W . The gauge field has ‘eaten’ the scalar fluctuation to become massive.

The type IIB action in the Einstein frame for the relevant fields, which yields the MN solution, is

$$S_{\text{IIB}} \propto \int d^{10}x \sqrt{g_{10}} \left[R - \frac{1}{2} e^{\Phi} |F_3|^2 \right]. \quad (15)$$

Plugging (12) and (14) into this action one obtains the action for the perturbation

$$S \propto - \int d^{10}x \sqrt{g_{10}} \left[\frac{N}{16} e^{\Phi/2} |F_2|^2 + \frac{3(16r^2+1)}{16r^2} W^2 \right], \quad (16)$$

where we have only shown the relevant terms for the discussion, dropping an overall constant term. Here $F_2 = dW$. One sees that a mass term for the gauge field W arises as a result of the spontaneous breaking of the gauge symmetry. Instead of W^2 in the action (16), one could have written $(D\lambda)^2$, with the appropriate covariant derivative $D\lambda = \partial\lambda - A$. Although the scalar field λ need not acquire a nonzero vacuum expectation value, its presence still implies spontaneous symmetry breakdown. This is because even the vacuum $\langle \lambda \rangle = 0$ is not invariant under the transformation $\lambda \rightarrow \lambda + \epsilon$.

There is no problem here with substituting the perturbation ansatz into the action. The gauge transformations (11) allow us to consider the perturbation in a gauge where $\lambda = 0$. Such perturbations about an isometry are known to be consistent, that is, the equations of motion obtained from the action (16) are the correct equations of motion for the perturbation. In the G_2 holonomy cases we consider below, checking consistency will be an important and nontrivial test of the scenario we present for spontaneous symmetry breaking.

2 Spontaneous symmetry breaking in G_2 backgrounds

In this section, we show the spontaneous symmetry breaking of a $U(1)$ fluctuation about G_2 holonomy backgrounds.

2.1 The G_2 holonomy metrics

All known G_2 holonomy metrics that are cohomogeneity one with principal orbits given by $SU(2) \times SU(2) = S^3 \times S^3$ admit the following form [19, 30]. At the end of this subsection we comment on the topology of these spaces and give some illustrations.

$$ds_{11}^2 = dx_4^2 + dr^2 + a(r)^2 [(\Sigma_1 + g(r)\sigma_1)^2 + (\Sigma_2 + g(r)\sigma_2)^2] + f(r)^2(\Sigma_3 + g_3(r)\sigma_3)^2 \\ + b(r)^2 [(\Sigma_1 - g(r)\sigma_1)^2 + (\Sigma_2 - g(r)\sigma_2)^2] + c(r)^2(\Sigma_3 - \sigma_3)^2, \quad (17)$$

where Σ_i, σ_i are left-invariant one-forms on two copies of $SU(2)$,

$$\begin{aligned} \sigma_1 &= \cos \psi_1 d\theta + \sin \psi_1 \sin \theta d\phi, \\ \sigma_2 &= -\sin \psi_1 d\theta + \cos \psi_1 \sin \theta d\phi, \\ \sigma_3 &= d\psi_1 + \cos \theta d\phi, \end{aligned} \quad (18)$$

where $0 \leq \theta \leq \pi$, $0 \leq \phi \leq 2\pi$, $0 \leq \psi_1 \leq 4\pi$, at least before including the \mathbb{Z}_N quotient, which we describe below. The definitions for Σ_i are analogous but with $(\theta, \phi, \psi_1) \rightarrow (\alpha, \beta, \psi_2)$. Note that here and throughout we work in units with the eleven dimensional Planck length $l_p = 1$.

The radial functions are defined through a constraint

$$g_3 = g^2 - \frac{c(a^2 - b^2)(1 - g^2)}{2a b f}, \quad (19)$$

and the following first order equations

$$\begin{aligned} \dot{a} &= \frac{c^2(a^2 - b^2) + [4a^2(a^2 - b^2) - c^2(5a^2 - b^2) - 4a b c f] g^2}{16a^2 b c g^2}, \\ \dot{b} &= -\frac{c^2(a^2 - b^2) + [4b^2(a^2 - b^2) + c^2(5b^2 - a^2) - 4a b c f] g^2}{16a b^2 c g^2}, \\ \dot{c} &= \frac{c^2 + (c^2 - 2a^2 - 2b^2) g^2}{4a b g^2}, \\ \dot{f} &= -\frac{(a^2 - b^2)[4a b f^2 g^2 - c(4a b c + a^2 f - b^2 f)(1 - g^2)]}{16a^3 b^3 g^2}, \\ \dot{g} &= -\frac{c(1 - g^2)}{4a b g}. \end{aligned} \quad (20)$$

These equations have two known integration constants

$$m = (b^2 - a^2)cg^2 + 2abfg^2g_3, \quad n = (b^2 - a^2)c + 2abf. \quad (21)$$

It is more convenient here to rewrite these metrics in a form that makes manifest the isometry along the coordinate $\gamma = \psi_1 + \psi_2$ and which completes the square in the tentative ‘massive isometry’ coordinate $\psi = \psi_1 - \psi_2$.

$$ds_{11}^2 = dx_4^2 + dr^2 + a^2 [(g^1)^2 + (g^2)^2] + b^2 [(g^3)^2 + (g^4)^2] + u^2(g^5)^2 + v^2(g^6)^2, \quad (22)$$

where we have introduced the new radial functions

$$u^2 = \frac{f^2(1 - g_3)^2}{4} + c^2, \quad v^2 = \frac{c^2 f^2(1 + g_3)^2}{4u^2}, \quad (23)$$

and also the terms

$$\begin{aligned} g^1 &= g(r)E^1 + E^3 \equiv -g(r)\sin\theta d\phi - \sin\psi d\alpha - \cos\psi \sin\alpha d\beta, \\ g^2 &= g(r)E^2 + E^4 \equiv g(r)d\theta + \cos\psi d\alpha - \sin\psi \sin\alpha d\beta, \\ g^3 &= g(r)E^1 - E^3, \\ g^4 &= g(r)E^2 - E^4, \\ g^5 &= d\psi + p(r)d\gamma + q(r)\cos\theta d\phi + s(r)\cos\alpha d\beta, \\ g^6 &= d\gamma + \cos\theta d\phi + \cos\alpha d\beta. \end{aligned} \quad (24)$$

Note that, although similar, E^3 and E^4 are not the standard conifold terms that were used to write a G_2 holonomy metric in [17]. The standard case would be recovered if we were to work with $\beta' = -\beta$. The further new radial functions introduced are

$$p = \frac{f^2(g_3^2 - 1)}{4u^2}, \quad q = \frac{2f^2g_3(g_3 - 1) + 4c^2}{4u^2}, \quad s = \frac{2f^2(g_3 - 1) - 4c^2}{4u^2}. \quad (25)$$

The angles have ranges $0 \leq \alpha, \theta \leq \pi$, $0 \leq \beta, \phi < 2\pi$, $0 \leq \psi < 4\pi$ and $0 \leq \gamma < 4\pi/N$. These variables make the ∂_γ isometry explicit. In the range of γ we have now included the quotient by \mathbb{Z}_N : $\gamma \sim \gamma + 4\pi/N$. The direction ∂_γ is usually thought of as the M theory circle. The quotient is important because on reduction to IIA, it will give N units of D6-brane flux. Classically, it is the only place in the M theory background where the N of the dual $SU(N)$ gauge theory appears.

General solutions to these equations are only known numerically. Two nonsingular exact solutions are known [15, 16, 17, 31], but will not be of particular interest here. We will work directly with the equations (20). These first order equations imply G_2 holonomy

of the metric. They further imply the set of second order equations for Ricci flatness of the metric (22). The second order equations allow more general, non-supersymmetric solutions. However, we will see that in considerations of consistency below, it is the first order equations that are required.

The equations (20) may be specialised to various cases that have been discussed in the literature. The unified form we use here allowed a classification of the known G_2 holonomy metrics with $SU(2) \times SU(2)$ principal orbits [19, 20]. Four families were discussed, denoted \mathbb{B}_7 [17, 32], \mathbb{C}_7 [33], $\widetilde{\mathbb{C}}_7$ [19] and \mathbb{D}_7 [18, 20]. The first fact that we shall use about these different families is that \mathbb{B}_7 and \mathbb{C}_7 metrics follow from making use of the consistent truncation of the first order equations $g(r) = 1$. However, we will ultimately argue that the \mathbb{D}_7 metrics are the gravitational dual of $\mathcal{N} = 1$ SYM field theory.

The \mathbb{B}_7 and \mathbb{D}_7 families will appear to be the most closely related to $\mathcal{N} = 1$ SYM. The \mathbb{B}_7 metrics have topology $S^3 \times (\mathbb{R}^4/\mathbb{Z}_N)$, and are singular, whilst the \mathbb{D}_7 metrics have nonsingular topology $S^3/\mathbb{Z}_N \times \mathbb{R}^4$. Both families have the same asymptotics. The topology is illustrated in the following figure.

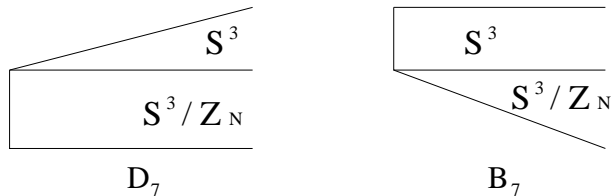


Figure 1: The \mathbb{B}_7 and \mathbb{D}_7 spaces.

The geometric differences between the distinct families may also be seen through a dimensionally reduced description in terms of four dimensional nonabelian monopoles and cosmic strings [34].

2.2 A fluctuation about the G_2 metrics

We will argue that the direction ∂_ψ is dual to the anomalous chiral symmetry of $\mathcal{N} = 1$ SYM gauge theory. This angle is not a $U(1)$ isometry of the metric. It does contain, however, a \mathbb{Z}_2 isometry of the metric, under $\psi \rightarrow \psi + 2\pi$. Therefore it is a natural candidate for the dual to the chiral symmetry, because chiral symmetry breaking should preserve a \mathbb{Z}_2 symmetry all the way into the IR. Another argument is that whilst not an isometry, this direction is still special in a way that will be formalised in a later section where we will call

such a direction a massive isometry. Further, ∂_ψ becomes an isometry at infinity.

Fluctuations about an isometry are well known to be described by gauge fields. Fluctuations about a massive isometry are described by gauge fields that obtain a mass through spontaneous symmetry breaking.

As described in the introduction, spontaneous symmetry breaking in supergravity solutions is the most natural way to break gauge symmetries that are dual to broken global symmetries in field theory. In particular, spontaneous symmetry breaking is known to provide the dual description to the chiral anomaly in the Klebanov-Strassler [27, 28] and Maldacena-Núñez, see introduction, solutions. A novel feature of the present case is that the background is purely gravitational.

Consider the following fluctuation about the metric (22) in the ∂_ψ direction

$$d\psi \rightarrow d\psi + A_a(x, r)dx^a, \quad \psi \rightarrow \psi + \lambda(x, r), \quad (26)$$

where the index a runs over the five coordinates t, x, y, z, r .

The coordinate transformation $\psi \rightarrow \psi + \epsilon(x, r)$ corresponds to the $U(1)$ gauge transformations

$$A \rightarrow A + d\epsilon, \quad \lambda \rightarrow \lambda + \epsilon. \quad (27)$$

Substituting this fluctuation into the eleven dimensional Einstein-Hilbert action, one obtains the following reduced action for the fluctuation

$$S \propto - \int d^4x dr \sqrt{g_7(r)} \left[\left(\frac{(g^2 - 1)^2 [2abf + c(a^2 - b^2)]^2}{64a^2b^2} + \frac{c^2}{4} \right) F_{ab}F^{ab} + \frac{[a^2 - b^2]^2}{4a^2b^2} D_a \lambda D^a \lambda \right], \quad (28)$$

where the field strength is as usual

$$F_{ab} = \partial_a A_b - \partial_b A_a, \quad (29)$$

the covariant derivative is that corresponding to the representation of λ

$$D_a \lambda = \partial_a \lambda - A_a, \quad (30)$$

and $g_7(r)$ is the determinant of the G_2 metric with the angular directions integrated out,

$$g_7(r) = a^2 b^2 c^2 g^4 [2abf(1 + g^2) + c(b^2 - a^2)(1 - g^2)]^2. \quad (31)$$

Indices are raised using the flat five-dimensional part of the metric (22), that is

$$ds_5^2 = dx_4^2 + dr^2. \quad (32)$$

It is clear that the action is invariant under the gauge transformations (27). The derivation of the action (28) only uses the equations for the radial functions (20) in order to set a cosmological term to zero, corresponding to Ricci-flatness of the eleven dimensional metric.

The reduced action is naturally interpreted as exhibiting spontaneous symmetry breaking, as we discussed towards the end of section 1.4. The scalar field λ need not acquire a nonzero vacuum expectation value. The gauge field A ‘eats’ the scalar field λ to become a gauge-invariant massive Stükelberg-Pauli vector field

$$W = A - d\lambda. \quad (33)$$

In terms of this field, the action becomes

$$S \propto - \int d^4x dr \sqrt{g_7(r)} \left[\left(\frac{(g^2 - 1)^2 [2abf + c(a^2 - b^2)]^2}{64a^2b^2} + \frac{c^2}{4} \right) F_{ab}F^{ab} + \frac{[a^2 - b^2]^2}{4a^2b^2} W_a W^a \right], \quad (34)$$

where now

$$F_{ab} = \partial_a W_b - \partial_b W_a (= \partial_a A_b - \partial_b A_a). \quad (35)$$

Thus vector fluctuations about the angle ∂_ψ do acquire a mass through spontaneous symmetry breaking, as required for the dual description of the anomalous chiral symmetry.

Let us note in passing that the action is greatly simplified in the case when $g(r) = 1$, corresponding to the \mathbb{B}_7 and \mathbb{C}_7 metrics. In this case

$$S_{g=1} \propto - \int d^4x dr \left[a^2 b^2 c^3 f F_{ab}F^{ab} + (a^2 - b^2)^2 c f W_a W^a \right]. \quad (36)$$

It is important to check that this reduction is consistent, that is, that the corresponding reduced equations of motion solve the initial eleven dimensional equations of motion. We do this in the following section. Beforehand, we make a few additional comments about why it was necessary to consider a non-isometric angle.

The most immediate reason, which we have already stated, is that we wish to exhibit spontaneous symmetry breaking. In the absence of p-form field strengths, this is not possible by perturbing an isometry. For a second reason, consider the continuous isometries of the metric: $SU(2) \times SU(2) \times U(1)$. The $U(1)$ isometry is just the M theory circle. If we would like the fluctuation to reduce to a fluctuation about the corresponding IIA background, then we should not perturb about this direction. The remaining $U(1)$ isometries are contained within $SU(2)$ s. But $\mathcal{N} = 1$ SYM does not have a global $SU(2)$ symmetry and in particular, the anomalous chiral $U(1)$ is not contained within an $SU(2)$.

A check of the idea of perturbing a non-isometric angle is as follows. Consider a similar background expected to be dual to a field theory without a chiral anomaly, and see whether

the symmetry breaking effect does not occur. Supersymmetric, purely gravitational backgrounds of M theory can be systematically constructed from considering D6-branes wrapping calibrated cycles in IIA backgrounds [35]. For example, N D6-branes wrapping an S^2 inside a nonsingular $K3$ manifold. The theory living on these D-branes is five dimensional $SU(N)$ SYM theory with eight real supercharges. This theory will not have an anomalous chiral symmetry, because it is five dimensional. The configuration is described in eleven dimensions by M theory on the resolved conifold with a singular \mathbb{Z}_N quotient [36]. The resolved conifold metric is asymptotically conical with base $T^{1,1}$. The $T^{1,1}$ contains an angle, ψ' , that is very similar to the ψ angle that we have just considered. In fact, if one reduces the G_2 metrics to IIA, then we also obtain an asymptotically conical geometry with base $T^{1,1}$, and with ψ the same angle in the $T^{1,1}$ as ψ' . An important difference between the two cases is that in the G_2 metrics, the direction ∂_ψ is not an isometry in the interior of the metric, whilst in the resolved conifold, $\partial_{\psi'}$ remains an isometry everywhere. Thus we find that the existence of an interesting non-isometric direction is correlated with the existence of the chiral anomaly.

The previous paragraph gives another reason to take ∂_ψ as the massive isometry. The shift $\psi \rightarrow \psi + \epsilon$ of the angle ψ in $T^{1,1}$ is generically dual, in string theory conifold backgrounds, to the field theory R-symmetry [37].

3 Consistency and inconsistency of fluctuations

In this section, we show that the fluctation about the G_2 backgrounds that we have just considered is asymptotically consistent in general and fully consistent in the \mathbb{B}_7 and \mathbb{C}_7 cases.

3.1 Equations of motion and consistency when $g(r) = 1$

Let us first calculate the equations of motion corresponding to the action we just derived (34). It is convenient to define the two functions

$$\begin{aligned} P(r)^2 &= \frac{(g^2 - 1)^2 [2abf + c(a^2 - b^2)]^2}{16a^2b^2} + c^2, \\ Q(r)^2 &= \frac{[a^2 - b^2]^2}{2a^2b^2}, \end{aligned} \tag{37}$$

So the action (34) is written

$$S \propto - \int d^4x dr \sqrt{g_7(r)} \left[\frac{P(r)^2}{4} F_{ab} F^{ab} + \frac{Q(r)^2}{2} W_a W^a \right]. \tag{38}$$

The equations of motion that follow from this action are

$$\begin{aligned}\partial_j F^{jr} &= \frac{Q^2}{P^2} W^r \\ \partial_r F^{ri} + \partial_j F^{ji} + \frac{\partial_r(\sqrt{g_7} P^2)}{\sqrt{g_7} P^2} F^{ri} &= \frac{Q^2}{P^2} W^i.\end{aligned}\quad (39)$$

We use indices $i, j \dots$ to run over the coordinates t, x, y, z , whilst as before $a, b \dots$ run over t, x, y, z, r . Now consider ∂_r of the first equation and subtract ∂_i of the second. The equation obtained is of type well known for massive vector fields

$$\partial_a W^a + \left[\frac{\partial_r(\sqrt{g_7} P^2)}{\sqrt{g_7} P^2} + \frac{P^2}{Q^2} \partial_r \left(\frac{Q^2}{P^2} \right) \right] W_r = 0. \quad (40)$$

We now need to see whether these equations solve the full eleven dimensional equations of motion. This is necessary in order for the fluctuations, which are eleven dimensional, to be on shell and hence physical.

As an illustration of when this works, consider first the case of the allowed truncation to $g = 1$. This corresponds to considering the \mathbb{B}_7 and \mathbb{C}_7 metrics. The equations of motion (39) in this case follow from the action (36) and the first order equations (20)

$$\begin{aligned}\partial_j F^{jr} &= \frac{(a^2 - b^2)^2}{2a^2 b^2 c^2} W^r \\ \partial_r F^{ri} + \partial_j F^{ji} - \frac{cf(b^2 - a^2) + 2ab(3a^2 + 3b^2 - c^2)}{4a^2 b^2 c} F^{ri} &= \frac{(a^2 - b^2)^2}{2a^2 b^2 c^2} W^i,\end{aligned}\quad (41)$$

while the divergence expression (40) becomes

$$\partial_a W^a - \frac{cf[(a^4 + b^4) + 6a^2 b^2] + 2ab[c^2(a^2 - b^2) + b^4 - a^4]}{4a^2 b^2 c(a^2 - b^2)} W_r = 0. \quad (42)$$

The equations of motion satisfied by the background metric in eleven dimensions (22) are

$$R_{\bar{u}\bar{v}} = 0, \quad (43)$$

where indices \bar{u}, \bar{v} run over the eleven tangent space coordinates. After including the fluctuation (26) in the metric, the Ricci tensor acquires the following nonzero components. It is important to note firstly that we are working to linear order in the fluctuation only and secondly that we work in a gauge where $\lambda = 0$, in which therefore $A = W$.

$$\delta R_{\bar{4}\bar{1}0} = \frac{c}{2} \left[-\partial_r F^{ri} - \partial_j F^{ji} + \frac{cf(b^2 - a^2) + 2ab(3a^2 + 3b^2 - c^2)}{4a^2 b^2 c} F^{ri} + \frac{(a^2 - b^2)^2}{2a^2 b^2 c^2} W^i \right], \quad (44)$$

$$\delta R_{\bar{5}\bar{1}0} = \frac{c}{2} \left[-\partial_j F^{jr} + \frac{(a^2 - b^2)^2}{2a^2 b^2 c^2} W^r \right], \quad (45)$$

$$\delta R_{\bar{6}\bar{9}} = -\delta R_{\bar{8}\bar{7}} = \frac{(a^2 - b^2)}{4ab} \left[\partial_a W^a - \frac{cf[(a^4 + b^4) + 6a^2b^2] + 2ab[c^2(a^2 - b^2) + b^4 - a^4]}{4a^2b^2c(a^2 - b^2)} W_r \right]. \quad (46)$$

The bars denote tangent space coordinates. The numbers correspond with the order that the vielbeins appear in the metric (22). Thus for example the $\bar{1}\bar{0}$ direction corresponds to the ug^5 vielbein. The equations for the radial functions (20) have been used.

It is immediate that the three equations of motion (41) and (42) imply the vanishing of $\delta R_{\bar{u}\bar{v}}$. Therefore the reduction is consistent to first order in the fluctuation. At various stages we used the first order equations (20). We needed specifically the first order equations for G_2 holonomy. The second order equations for Ricci flatness of the metric (22) would not have been sufficient, although they would have been sufficient to derive the action (34). Thus, special holonomy plays an important role in the consistency of the fluctuation.

3.2 Asymptotic consistency for general G_2 metrics

Now let us consider the more general G_2 holonomy metrics, which do not have $g(r) = 1$. Now things do not work so clearly. To show that the perturbation is in fact inconsistent in general, we will substitute the equations of motion (39) and (40) into the fluctuation of the Ricci tensor $\delta R_{\bar{u}\bar{v}}$ and find that it is not zero, and that the remaining nonzero terms cannot be set to zero by the equations of motion.

Concretely, after using the equations of motion, the fluctuation of the Ricci tensor has the following nonzero components

$$\delta R_{\bar{i}\bar{1}\bar{1}} = (g^2 - 1) \left(\frac{(b^2 - a^2)^2[c(a^2 - b^2) + 2abf]}{\sqrt{C(r)}} W_{\bar{i}} + \frac{B(r)}{\sqrt{C(r)}} F_{\bar{i}r} \right), \quad (47)$$

$$\delta R_{\bar{5}\bar{1}\bar{1}} = (g^2 - 1) \frac{(b^2 - a^2)^2[c(a^2 - b^2) + 2abf]}{\sqrt{C(r)}} W^r, \quad (48)$$

where $B(r), C(r)$ are complicated functions of the radial functions a, b, c, f, g . Their precise form does not seem to be illuminating, so we don't include them here. The only common factor amongst these terms is $g^2 - 1$. These terms cannot be set to zero using the reduced equations of motion. Therefore, the fluctuation is inconsistent unless $g^2 = 1$.

At first sight, this seems problematic for the \mathbb{D}_7 and $\widetilde{\mathbb{C}}_7$ metrics which have $g \neq 1$. However, all the families of G_2 holonomy metrics described by (22) have the same asymptotics as $r \rightarrow \infty$. In particular $g^2 \rightarrow 1$ asymptotically, which suggests a notion of asymptotic consistency.

In fact, whilst all the metrics have the same leading order asymptotics, there are two possible forms at subleading orders. The first is that of the \mathbb{B}_7 and \mathbb{C}_7 metrics and is

$$\begin{aligned}
a(r) &\sim \frac{r}{\sqrt{12}} + \frac{\sqrt{3}}{2} \left(q - \frac{R}{2} \right) + \frac{3\sqrt{3}R^2}{4r} + \dots, \\
b(r) &\sim -\frac{r}{\sqrt{12}} - \frac{\sqrt{3}}{2} \left(q + \frac{R}{2} \right) - \frac{3\sqrt{3}R^2}{4r} + \dots, \\
c(r) &\sim \frac{r}{3} + q + \frac{3R^2}{r} + \dots, \\
f(r) &\sim R - \frac{9R^3}{r^2} + \dots, \\
g(r) &= 1,
\end{aligned} \tag{49}$$

where R, q are free constants. These solutions have $n = m$, where n and m are the integration constants of (21).

The second form of subleading asymptotics, which is that of the \mathbb{D}_7 and $\widetilde{\mathbb{C}}_7$ metrics, is

$$\begin{aligned}
a(r) &\sim \frac{r}{\sqrt{12}} + \frac{\sqrt{3}}{2} \left(q - \frac{R}{2} \right) + \frac{\sqrt{3}(6R^3 + m)}{8Rr} + \dots, \\
b(r) &\sim -\frac{r}{\sqrt{12}} - \frac{\sqrt{3}}{2} \left(q + \frac{R}{2} \right) - \frac{\sqrt{3}(6R^3 + m)}{8Rr} + \dots, \\
c(r) &\sim \frac{r}{3} + q + \frac{3R^2}{r} + \dots, \\
f(r) &\sim R + \frac{3m - 9R^3}{r^2} + \dots, \\
g(r) &\sim 1 - \frac{3m}{2Rr^2} + \dots,
\end{aligned} \tag{50}$$

where m is a new constant. For the \mathbb{D}_7 metrics, m is just the first integration constant in (21). The \mathbb{D}_7 metrics further have $n = 0$, where n is the other integration constant of (21).

With these expansions we may study the question of consistency at large r . If one calculates the Ricci tensor at large r in tangent space coordinates one finds the following, in a hopefully obvious notation,

$$\begin{aligned}
\delta R_{\bar{i}\bar{1}0} &\sim \mathcal{O}(r)\partial F + \mathcal{O}(1)F + \mathcal{O}(r^{-3})W, \\
\delta R_{\bar{5}\bar{1}0} &\sim \mathcal{O}(r)\partial F + \mathcal{O}(r^{-3})W, \\
\delta R_{\bar{i}\bar{1}1} &\sim \mathcal{O}(r^{-3})F + \mathcal{O}(r^{-6})W, \\
\delta R_{\bar{5}\bar{1}1} &\sim \mathcal{O}(r^{-6})W, \\
\delta R_{\bar{6}\bar{9}} &\sim \delta R_{\bar{7}\bar{8}} \sim \mathcal{O}(r^{-1})\partial W + \mathcal{O}(r^{-2})W.
\end{aligned} \tag{51}$$

These results show that the inconsistent components, $\delta R_{\bar{i}\bar{1}1}, \delta R_{\bar{5}\bar{1}1}$ have coefficients of $W_{\bar{i}}$ and $F_{\bar{i}\bar{r}}$ that are suppressed by three powers of r compared with the consistent terms. Thus

at large r the fluctuation solves the full eleven dimensional equations of motion to a good approximation.

Below we will suggest that, as is common in gauge-gravity dualities, large radius corresponds to the UV of the dual field theory. The physical interpretation of the asymptotic consistency is that we have found the correct gravitational description of the field theory symmetry current in the UV but that a more complicated description is required in the IR. This is not necessarily surprising, given that as well as being anomalously broken to \mathbb{Z}_{2N} , the chiral symmetry is further spontaneously broken in the IR to \mathbb{Z}_2 .

The asymptotic consistency of the fluctuation is an important test of our proposal for spontaneous symmetry breaking. We will see in the last section below that such simple vector fluctuations about a generic non-isometric direction are not asymptotically consistent. Thus the ∂_ψ direction is special in this regard.

4 Comparison with generic five dimensional results

In this section we dimensionally reduce the action for our perturbation to five dimensions. We obtain a massive vector field with a mass that is asymptotically precisely that predicted on general grounds for vector fields dual to the chiral current.

4.1 A five dimensional prediction

In studying holographic renormalisation group flows, a general prediction was made for the mass of a vector field dual to a broken chiral symmetry [25]. Five dimensional backgrounds of the form

$$ds_5^2 \equiv g_{ab} dx^a dx^b = dq^2 + e^{2T(q)} dx_4^2, \quad (52)$$

were considered. It was argued that the transverse part of the vector field, V , associated with the chiral symmetry, would have an action, with the metric in the Einstein frame, given by

$$S = - \int d^4x dq \sqrt{-g} \left[\frac{1}{4} G_{ab} G^{ab} - \frac{1}{2} 2 \frac{\partial^2 T}{\partial q^2} V_a V^a \right], \quad (53)$$

where $G = dV$. In particular, the mass is $m^2 = -2\partial_q \partial_q T$. The reason one only studies the transverse part, with $V^q = \nabla_a V^a = 0$, is that in general one does a gauge field redefinition to obtain the gauge-kinetic term with canonical normalisation. The action for the V_q and divergence components become more complicated because of explicit q -dependence in the action.

We would thus like to know whether a dimensionally reduced version of our perturbation to five dimensions may be put into this form.

4.2 Dimensional reduction of the G_2 fluctuation

Start with the action (34) written as

$$S = - \int d^4x dr \sqrt{-g_5} \sqrt{g_6} \left[\frac{1}{4} P(r)^2 F_{ab} F^{ab} + \frac{1}{2} Q(r)^2 W_a W^a \right]. \quad (54)$$

The five dimensional metric is just

$$ds_5^2 = dr^2 + dx_4^2, \quad (55)$$

and the determinant g_6 is obtained by integrating out the angles of the G_2 holonomy metric from the determinant (31) where we have now included the numerical factor from integrating out the angles:

$$g_6 = \frac{2^{10} \pi^4}{N} a^2 b^2 c^2 g^4 [2abf(1+g^2) + c(b^2 - a^2)(1-g^2)]^2. \quad (56)$$

We only work with the transverse components, so again $W^r = \partial_a W^a = 0$. Recall that transverse and scalar components do not mix at a linearised level, so this truncation is consistent. In order to compare with the result we quoted in the previous subsection, we need to be in the Einstein frame. Thus we define

$$e^{-6\phi(r)} = g_6, \quad (57)$$

and define a rescaled metric by

$$d\tilde{s}_5^2 = e^{-2\phi(r)} ds_5^2. \quad (58)$$

In terms of this metric, the action now becomes

$$S = - \int d^4x dr \sqrt{-\tilde{g}_5} \left[\frac{1}{4} P(r)^2 F_{\tilde{g}}^2 e^{-2\phi} + \frac{1}{2} Q(r)^2 W_{\tilde{g}}^2 \right], \quad (59)$$

where the \tilde{g} subscript denotes that the indices are contracted with the rescaled metric, \tilde{g}_5 . We must now perform a field redefinition to obtain a canonically normalised kinetic term for the gauge field. Defining a new field, V_a , by

$$W_a = \frac{e^\phi}{P} V_a, \quad (60)$$

the action takes the form

$$S = - \int d^4x dr \sqrt{-\tilde{g}_5} \left[\frac{1}{4} G_{\tilde{g}}^2 + \frac{1}{2} m(r)^2 V_{\tilde{g}}^2 \right], \quad (61)$$

where again $G = dV$ and the mass is given by

$$m(r)^2 = e^{2\phi} \left[-\phi'' + 2(\phi')^2 - 3\frac{P'\phi'}{P} + \frac{P''}{P} + \frac{Q^2}{P^2} \right], \quad (62)$$

where the primes denote differentiation with respect to r . The action is now in the same form as the action (53) with which we wish to compare the mass term.

We can use the asymptotic expressions for the radial functions (49) and (50) to find that asymptotically (62) becomes

$$m(r)^2 = e^{2\phi} \left[\frac{80}{9r^2} + \mathcal{O}\left(\frac{R^2}{r^4}\right) \right]. \quad (63)$$

Now compare this with the prediction from the previous subsection. Note that from comparing (52) and (58) we see that $dq = e^{-\phi(r)} dr$ and $T(q) = -\phi(r)$. Thus:

$$m_{\text{prediction}}^2 = -2\frac{\partial^2 T}{\partial q^2} = 2e^{2\phi} [\phi'' + (\phi')^2], \quad (64)$$

which is asymptotically

$$m_{\text{prediction}}^2 = e^{2\phi} \left[\frac{80}{9r^2} + \mathcal{O}\left(\frac{R^2}{r^4}\right) \right]. \quad (65)$$

Thus we have found exact agreement to leading order with the general prediction of [25]. The agreement does not continue to subleading orders.

The leading order matching we have just described does not depend on the eleven dimensional mass, Q^2/P^2 , which turns out to be subleading in (62). It is important to emphasise that the five dimensional mass we are considering here has a different character to the mass in previous sections. Even an isometry will generically have a mass term if one rescales the gauge field to get canonical normalisation. The nontrivial point of the previous section is that when one has a massive isometry, the mass generation is from spontaneous symmetry breaking. The canonical normalisation was not the appropriate normalisation to use in that context.

4.3 Generic agreement to leading order

The agreement we have just found supports the setup we are presenting. Unfortunately, the particular direction we are perturbing is not the unique direction that would have resulted in an agreement. More concretely, from comparing (62) and (64) one sees that the following conditions are sufficient to imply agreement to leading order as $r \rightarrow \infty$:

$$\begin{aligned} \frac{P''}{P} \text{ and } \frac{Q^2}{P^2} &\ll \phi'', \\ P &\propto \frac{1}{\phi'}. \end{aligned} \quad (66)$$

From (57) we see that whenever the determinant g_6 is polynomial to leading order as $r \rightarrow \infty$, then $1/\phi' \sim r$. This is the case in all the G_2 metrics we have been considering. Now recall that the G_2 metrics are asymptotically locally conical, and that $P(r)$ is just the radial function in the metric in front of the angle we perturb. Thus, for a generic angle in the metric, except ∂_γ which stabilises at infinity, we will have $P(r) \sim r$. The second condition in (66) is therefore satisfied. This also implies that the first half of the first condition, $P''/P \ll \phi''$ is generically satisfied. The remaining condition $Q^2/P^2 \ll \phi''$ would need to be checked for every consistent perturbation, but it will certainly be satisfied by isometries, which have no mass in eleven dimensions: $Q_{\text{isometry}} = 0$.

Therefore, if we had perturbed any of the isometries in the G_2 holonomy metric, besides the M-theory $U(1)$, we would also have found agreement to leading order. At the end of section 2.2 we gave a few reasons however for why isometries of the metric were unlikely to be relevant for the dual description of the chiral anomaly.

The conclusion of this section seems to be that although one obtains encouraging results from a five dimensional perspective, the dimensionally reduced picture needs to be better understood. In particular, it would be nice to embed the fluctuation into a five dimensional supergravity theory. This would enable one to see whether the vector fluctuation is in the same supersymmetry multiplet as the metric, as is usually the case for R-symmetry gauge fields. This is because the R-symmetry current and the energy momentum tensor of the field theory are in the same anomaly multiplet. One route towards this could be via eight dimensional supergravities, which have been used to construct G_2 metrics [36, 38, 39].

4.4 Agreement for the Maldacena-Núñez background

We can easily apply the methods we have just developed to the fluctuation about the MN background that we considered in section 1.4. However, first we should note that at large r , the D5-brane solutions we described has diverging dilaton coupling (5). Therefore, if we wish not to deal with large stringy corrections, we should use the S dual solution describing NS5-branes. In the Einstein frame solution (4) this simply corresponds to letting $\Phi \rightarrow -\Phi$ in the metric. Further, the dilaton now has asymptotic behaviour

$$e^{2\Phi} = e^{2\Phi_0} \frac{4\sqrt{r}}{e^{2r}}. \quad (67)$$

The ten dimensional action for the fluctuation is now given by

$$S \propto - \int d^{10}x \sqrt{g_{10}} \left[\frac{N}{16} e^{-\Phi/2} |F_2|^2 + \frac{3(16r^2 + 1)}{16r^2} e^{2\Phi} W^2 \right]. \quad (68)$$

This action may be reduced to five dimensions and the metric rescaled to be in the five dimensional Einstein frame in essentially the same way as in the previous two subsections.

The generic prediction and the dimensionally reduced masses agree to leading order, and are given by

$$m(r)^2 \propto r^{2/3} e^{-4\Phi/3} (\Phi')^2 + \dots \propto r^{1/3} e^{r/3}. \quad (69)$$

If one had not S dualised, then the leading coefficient of the masses would not have agreed, showing that stringy corrections can upset the matching. This is in contrast to the G_2 holonomy backgrounds, where one can use the same \mathbb{D}_7 metric at large and small r .

5 Wrapped D5 and D6-branes, and decoupling

The G_2 metrics we are studying are related to wrapped D6-branes. This section considers the decoupling of Kaluza-Klein modes and gravity from the dual field theory. Comparisons with the Maldacena-Núñez solution of wrapped D5-branes are made. We discuss the breaking $U(1) \rightarrow \mathbb{Z}_{2N}$.

The G_2 manifolds we are discussing are M theory lifts of supersymmetric configurations of N D6-branes wrapped on a three-cycle in a Calabi-Yau manifold [10, 11, 14]. The M theory circle is the $U(1)$ isometry ∂_γ . The low energy theory living on the wrapped D6-branes is $\mathcal{N} = 1$ SYM. Although the first of the new G_2 metrics constructed in this context were the \mathbb{B}_7 metrics [17], these metrics have an ADE singularity at the origin and do not describe the full back-reaction of the branes on the geometry.

The subsequently discovered \mathbb{D}_7 [20, 18] metrics are not only well-behaved at the origin, but have the topology expected from studies of the M theory flop [11, 14, 21, 22] to describe the confining IR of $\mathcal{N} = 1$ gauge theory. Recall that the topology of the \mathbb{D}_7 metrics is $S^3/\mathbb{Z}_N \times \mathbb{R}^4$. At large radial direction, the \mathbb{D}_7 metrics have the same asymptotics as the \mathbb{B}_7 metrics, see equations (49) and (50) above, and thus should also capture some of the UV physics, as we are arguing in this paper.

The last two paragraphs suggest that the \mathbb{D}_7 metrics are the dual geometries to wrapped D6-branes in Type IIA string theory, in the same way that the Maldacena-Núñez solution is the dual geometry to wrapped D5-branes in type IIB theory. The IIA reduction of the \mathbb{D}_7 metrics is the resolved conifold with N units of RR flux through the noncollapsed S^2 . Within the geometric transition perspective introduced in [10], this expectation of duality is very natural.

The following discussion about decoupling is applicable to both the wrapped D5 and D6 configurations. For most of the time, we shall discuss only the D6 case explicitly.

There are two issues of decoupling when considering wrapped branes. One is the decoupling of Kaluza-Klein modes on the wrapped cycle and the other is the decoupling of gravity from field theory in the ‘field theory limit’.

5.1 Kaluza-Klein modes

First consider the Kaluza-Klein modes. The fact that the wrapping of the branes involves twisting suggests that the somewhat naïve analysis we are about to present may not be the whole story. We will say few words about this below. For this discussion only, we work explicitly with factors of α' .

The mass scale associated with the Kaluza-Klein modes should be

$$\Lambda_{\text{KK}}^3 \equiv M_{\text{KK}}^3 \sim \frac{1}{\text{Vol} S^3}, \quad (70)$$

where S^3 is the cycle in the Calabi-Yau wrapped by the branes. There is no a priori reason to identify this cycle with any particular cycle in the backreacted G_2 holonomy geometry.

We would like to compare this scale with the characteristic mass scale of the field theory generated through dimensional transmutation. This is defined to be the energy scale at which the one loop coupling diverges. Up to the Kaluza-Klein mass scale, the theory is effectively four dimensional and the coupling constant will follow the $\mathcal{N} = 1$ beta function. In particular, we may use the logarithmic one loop running of the effective coupling at the energy scale Λ_{KK} to obtain the following expression for the super Yang-Mills energy scale

$$\Lambda_{\text{SYM}}^3 \sim \Lambda_{\text{KK}}^3 e^{\frac{-8\pi^2}{g_{\text{eff},4}^2(\Lambda_{\text{KK}})}}. \quad (71)$$

Now, at the energy scale Λ_{KK} we should match the dimensionless four dimensional ‘t Hooft effective coupling with the dimensionless seven dimensional effective coupling $g_{\text{eff},4}^2(\Lambda_{\text{KK}}) = g_{\text{eff},7}^2(\Lambda_{\text{KK}})$. Further, on dimensional grounds, we know how to relate the dimensionless effective coupling to the dimensionful Yang-Mills coupling that appears in the action $g_{\text{eff},7}^2(\mu) = \mu^3 N g_{\text{YM},7}^2$. In particular, this implies $g_{\text{eff},7}^2(\Lambda_{\text{KK}}) = N g_{\text{YM},7}^2 / \text{Vol} S^3$. Finally, we can relate the Yang-Mills coupling to string theory quantities and hence to the M theory Planck length in the standard way [40]: $g_{\text{YM},7}^2 = (2\pi)^4 g_s \alpha'^{3/2} = (2\pi)^4 l_p^3$. Combining all these statements allows us to re-express (71) as

$$\Lambda_{\text{SYM}}^3 \sim \Lambda_{\text{KK}}^3 e^{\frac{-\text{Vol} S^3}{N^2 \pi^2 l_p^3}}. \quad (72)$$

To decouple the Kaluza-Klein modes, we would like to take

$$\Lambda_{\text{SYM}} \ll \Lambda_{\text{KK}}, \quad (73)$$

which requires

$$\text{Vol}S^3 \gg N2\pi^2 l_p^3. \quad (74)$$

This is always possible in principle, the question is whether it is compatible with the regime of validity of the gravity dual.

On general grounds, we should not expect a perturbative regime in field theory to overlap with a regime in which a supergravity description is valid, else the conjectured duality would be falsified by the fact that the two theories are manifestly different. An example of this is the well-studied case of near horizon geometries of flat D-branes, where the scalar curvature in the string frame satisfies [40]

$$\alpha' R_{\text{str}} \sim \frac{1}{g_{\text{eff}}}, \quad (75)$$

so clearly perturbative field theory and small gravitational curvatures are incompatible. In general the dependence of g_{eff} on the energy scale, μ , translates into a radial, r , dependence of $\alpha' R$. Typically, small r will be the IR of the field theory, and large r will be the UV.

It seems plausible that to have a valid gravitational description, the dual effective coupling should be large. The minimum of the effective coupling is at Λ_{KK} because above this scale it will increase according to the seven dimensional relation $g_{\text{eff},7}^2(\mu) = \mu^3 N g_{\text{YM},7}^2$, and below this scale it will increase due to the logarithmic running of the four dimensional coupling. This is illustrated in the following figure.

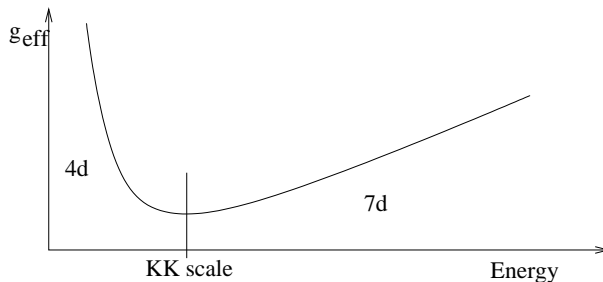


Figure 2: The running of the effective coupling.

We have just seen that $g_{\text{eff},4}^2(\Lambda_{\text{KK}}) = \frac{N16\pi^4 l_p^3}{\text{Vol}S^3}$. Therefore the regime in which the effective coupling is large, $\text{Vol}S^3 \ll N16\pi^4 l_p^3$, has no overlap with the decoupling regime (74). This is the standard problem of strong coupling gauge theory duals: in order to obtain the dual, one introduces a new scale, in this case the Kaluza-Klein scale, which then cannot be decoupled

from the field theory scale. Note that in our argument we have not made any assumptions relating certain cycles within the gravitational background with the initial cycle the branes wrap, as is often done in discussions of the Maldacena-Núñez background, for example [9, 41]. Instead we have made the assumption that the gravitational background should not contain a regime dual to a weakly coupled field theory.

As we noted above, the analysis we have just presented may be too naïve. In particular, we have not taken into account the fact that the theory living on wrapped D-branes is partially twisted [42] in order to be supersymmetric. The degrees of freedom along the wrapped directions are topological. One might expect this fact to modify the role of the Kaluza-Klein scale in the theory and the running of the coupling above the Kaluza-Klein scale.

So far we have discussed general expectations from the renormalisation group flow in field theory. Now we turn to the other side of the duality and calculate curvatures and string couplings for the gravity backgrounds. It is not difficult to explicitly calculate the Ricci scalar and dilaton for the string frame IIA backgrounds resulting from dimensional reduction of the \mathbb{D}_7 metrics [18]. One finds that the Ricci scalar decreases away from the origin. As we are now discussing the background itself, we return to units with $\alpha' = 1$. One finds

$$R_{\text{str}, r=0} \sim \mathcal{O}\left(\frac{N}{L^2}\right), \quad (76)$$

where here L is the radius of the S^2 that does not collapse at the origin in the IIA metrics, and carries N units of Ramond-Ramond flux. On the other hand, the dilaton increases away from the origin. For a typical value of the nontrivial parameter of the \mathbb{D}_7 metrics, i.e. not the scaling parameter, (50), the dilaton coupling remains the same order of magnitude. That is

$$g_s \equiv e^\Phi \sim \mathcal{O}\left(\frac{L}{N^{3/2}}\right). \quad (77)$$

The last two formulae imply that

$$R_{\text{str}, r=0} \sim \frac{1}{g_s^2 N^2}, \quad (78)$$

which might be compared with the result for the Maldacena-Núñez solution: $R_{\text{str}} \sim 1/g_s N$.

Thus small curvature requires $g_s N \gg 1$. This will be consistent with small string coupling if we take N large. Unlike the Maldacena-Núñez case, the dilaton remains finite at infinity, and so the lift to M theory is not always forced.

The G_2 manifold itself is of course Ricci flat, but one can calculate the Riemann squared curvature invariant. This is seen to decrease to zero away from the origin. At the origin

one has

$$R^{abcd}R_{abcd}\Big|_{\mathbb{D}_7, r=0} \sim \mathcal{O}\left(\frac{1}{\tilde{L}^4}\right), \quad (79)$$

where \tilde{L} is the radius of the noncollapsed S^3 in the \mathbb{D}_7 metric. We can then safely take \tilde{L} large to avoid large M-theoretic corrections to the background.

5.2 Gravitational modes

The decoupling limit of D6-branes is well-known to be problematic [40]. The $\alpha' \rightarrow 0$ limit is taken with $g_{\text{YM},7}$ held fixed. As we have just recalled, $g_{\text{YM},7}^2 = (2\pi)^4 g_s \alpha'^{3/2} = (2\pi)^4 l_p^3$, which implies a finite eleven dimensional Planck length and hence the non-decoupling of gravity. Further, the characteristic mass scale of gravitational contamination in the field theory is $1/l_p$. The discussion of the previous subsection shows that we cannot decouple Λ_{KK} and $1/l_p$ within the regime of gravitational description. Therefore the gravitational scale also cannot be decoupled from Λ_{SYM} .

Two features that occur in the M theory lift of flat D6 branes, and which are related to the non-decoupling of gravity [40] also occur in the G_2 metrics. Firstly, $R^{abcd}R_{abcd}$ goes to zero asymptotically. Secondly, the curvatures of the metric do not depend on N at all.

The analogous problem in the D5-branes wrapping an S^2 of the Maldacena-Núñez solution is the decoupling of the little string scale. However, the D6-brane case is worse because the non-decoupling of gravity means that the field does not have UV completion other than M theory itself.

It is remarkable that despite such problems with decoupling, dualities involving wrapped branes are able to reproduce features of pure $\mathcal{N} = 1$ Super Yang Mills theories. Most features are qualitative or of a topological character [9], so it is particularly mysterious that the perturbative beta function for $\mathcal{N} = 1$ Super Yang Mills was recently calculated from the Maldacena-Núñez solution [29, 43, 44]. Perhaps the success is related to the fact, as we mentioned previously, that the degrees of freedom in the wrapped directions of the brane are described by a topological field theory.

Another quantitative matching that has been possible in other other dualities is the manifestation of the anomalous breaking $U(1)_R \rightarrow \mathbb{Z}_{2N}$ in the gravity side of the duality. We turn to this point next, and see that whilst the MN case works very straightforwardly, the G_2 case does not. This may be related to the different nature of the non-decoupling in the two models, little string modes as opposed to full M theory modes.

5.3 $U(1) \rightarrow \mathbb{Z}_{2N}$ in the Maldacena-Núñez background

Let us review how this mechanism works in the MN solution [9]. Recall from the introduction that the metric (4) has a symmetry under

$$\psi \rightarrow \psi + \epsilon. \quad (80)$$

The metric is invariant under this constant shift. It will be a symmetry of the background if the shift (80) is also a gauge transformation of the RR potential.

In fact, the RR potential (8) transforms as

$$C_2 \rightarrow C_2 + \frac{\epsilon}{4} N \omega_2 \quad (81)$$

under (9). Classically, this is a gauge transformation because $d\omega_2 = 0$. However, there are quantum mechanical complications because although ω is closed, it is not exact. This is possible because of the topology at infinity $H^2(S^2 \times S^3, \mathbb{Z}) = \mathbb{Z}$. To see the effect of this, consider a probe instantonic D1-brane that is coupled to C_2 with the following contribution to the partition function

$$Z_{D1}[C_2] \sim e^{\frac{i}{2\pi} \int_{S^2} C_2}, \quad (82)$$

where S^2 has ω_2 as volume form. Z_{D1} is required to be invariant under the gauge transformations, $C_2 \rightarrow C_2 + \lambda_2$. While this invariance is trivial for an exact λ_2 , one should also require the invariance under ‘large’ gauge transformations where λ_2 is proportional to the volume form on S^2 , that is $\lambda_2 = c \omega_2$. Then invariance of (82) fixes c to be an arbitrary integer multiple of $\frac{\pi}{2}$. Thus we conclude that as long as the shift of C_2 under the chiral transformation (81) is equivalent to a “large” gauge transformation $C_2 \rightarrow C_2 + n \frac{\pi}{2} \omega_2$, the chiral transformation will be a symmetry of the whole background. This condition fixes

$$\epsilon = \frac{2n\pi}{N}, \quad (83)$$

where n is an arbitrary integer. Since ϵ is defined modulo 4π we see that there are $2N$ discrete values of the gauge transformation that is preserved by the background. Thus the $U(1)$ symmetry is broken to \mathbb{Z}_{2N} .

5.4 $U(1) \rightarrow \mathbb{Z}_{2N}$ in G_2 holonomy backgrounds?

Can this argument be adapted to the G_2 holonomy case we are studying? We will show that, classically at least, this is not possible. The natural adaptation of the logic would be to have a nonvanishing three-form field C such that under a shift in the massive isometry

direction, $\psi \rightarrow \psi + \epsilon$, one had $\delta C \sim N\epsilon\omega_3$. Where ω_3 would be the volume form of one of the S^3 s in the G_2 metric. The shift ϵ would then become quantised using the same argument as before, but with instantonic M theory membranes instead of D1-branes.

However, any C -field that we add classically must have $G = dC = 0$ in order not to backreact and spoil the G_2 holonomy of the metric. The fact that C must be closed means that the periods of C are invariant under homology. This will not allow the desired periods of the form $\int C \sim N\psi$, because two cycles at different values of ψ are homologous but would have different periods, contradicting our previous statement.

The only way of introducing a ψ dependence into C without having the periods proportional to ψ is to integrate over ψ also. This would require a term like $C \sim \psi d\psi \wedge \omega_2$, for some closed two-form ω_2 . Such a term will also not work, because as $\psi \rightarrow \psi + \epsilon$ then $\delta C = \epsilon d\psi \wedge \omega_2 = d(\epsilon\psi \wedge \omega_2)$. Thus the change in C is exact, and will not cause a change in phase of the membrane partition function.

One should then investigate whether quantum mechanical effects may alter the situation. Atiyah and Witten have shown [12] that M theoretic corrections to G_2 holonomy vacua are under some control, and in particular that nonvanishing four-form fluxes $\int G$ are induced. A naïve application of the results in [12], which use the old G_2 metrics, does not look as promising as it might. Let us sketch why.

Atiyah and Witten parameterise the space of asymptotically G_2 vacua in terms of three complex variables η_1, η_2, η_3 . The only property of these variables that we will require is that

$$\alpha_i \equiv \arg \eta_i = \int_{D_i} C + \mu(D_i)\pi, \quad (84)$$

where D_i is a three-cycle of the geometry at infinity. The second term, $\mu(D_i)$ is a topological correction due a membrane fermion anomaly, it can be 0 or 1. There are three such cycles if one thinks of the orbits of the cohomogeneity one G_2 metrics as

$$\frac{S^3}{\mathbb{Z}_N} \times S^3 = \frac{SU(2)/\mathbb{Z}_N \times SU(2) \times SU(2)}{SU(2)}. \quad (85)$$

The cycles satisfy the constraint $ND_1 + D_2 + D_3 = 0$. A major result of [12] is that the space of supersymmetric vacua that are asymptotically G_2 , including M theory corrections, is given by the following equations

$$\begin{aligned} \eta_2 &= \eta_1^{-N}(\eta_1 - 1)^N, \\ \eta_3 &= (1 - \eta_1)^{-N}. \end{aligned} \quad (86)$$

These equations parameterise a one complex dimensional space of supersymmetric vacua that interpolates smoothly between three classical regimes. These classical regimes are

distinguished by which of the three cycles at infinity, D_i , is filled in by the full G_2 metric. In the \mathbb{D}_7 metrics, the cycle with the \mathbb{Z}_N quotient does not collapse in the interior, so it is similar to the classical points in the space of vacua P_2 and P_3 considered by [12], in which D_2 and D_3 collapse respectively. Without loss of generality we take it to be P_3 . Note that [12] did not consider the \mathbb{D}_7 metrics, as they were working with the older G_2 holonomy metrics. In this sense the present analysis is only preliminary, and could change substantially if one redid the calculations of Atiyah and Witten with the new metrics.

In the classical small curvature limit, that is, for the point P_3 , one has

$$\int_{D_3} C = \int_{\mathbb{R}^4} G = 0. \quad (87)$$

One can also show [12] that $\mu(D_3) = 0$ in this case, so $\alpha_3 = 0$. Further, it turns out that as we tend towards P_3 , then $|\eta_1| \rightarrow 0$. The equations (86) now imply that in the classical limit

$$\alpha_2 = -N\alpha_1 + N\pi. \quad (88)$$

The periods of the C field are also related because $ND_1 + D_2 + D_3 = 0$ implies that

$$\int_{D_2} C = -N \int_{D_1} C. \quad (89)$$

As argued above, the classical situation is not interesting because the C field cannot have the required ψ dependence. The quantum corrections to the C field may be calculated from (86). Take $\eta_1 = |\eta_1|e^{i\Theta}$, with $|\eta_1| \ll 1$ so that we are near the classical limit. Then the quantum corrected periods to leading order away from the classical point P_3 are

$$\begin{aligned} \int_{D_1} C &= \Theta, \\ \int_{D_2} C &= -N\Theta - N|\eta_1| \sin \Theta, \\ \int_{D_3} C &= N|\eta_1| \sin \Theta. \end{aligned} \quad (90)$$

Whilst the appearance of factors of N might appear promising, these formulae do not allow the desired effect. Firstly, $|\eta_1|$ and Θ are parameterising the space of vacua, so they cannot depend on the coordinate ψ , they are constant for a given background. Secondly, the equations (90) tie the quantum corrected periods to the classical periods. Thus the quantum corrections cannot change under $\psi \rightarrow \psi + \epsilon$ without the classical periods also changing. But we established above that the classical periods could not change.

Nonetheless, to test this properly, one should extend the full analysis to the newer G_2 metrics that we have been studying. This seems like a worthwhile calculation to carry out in any case.

A remnant of the $U(1) \rightarrow \mathbb{Z}_{2N}$ breaking is seen in the existence of N vacua in the IR, where $\mathbb{Z}_{2N} \rightarrow \mathbb{Z}_2$. The topological studies of confining strings [14, 21] and domain walls [14, 22] in the IR of the G_2 metrics support this picture.

An alternative argument for seeing $U(1) \rightarrow \mathbb{Z}_{2N}$ in G_2 backgrounds was suggested by [13]. This involved a four-form characteristic class $\lambda = p_1(G_2)/2$. It seems unlikely that this argument can be applied here because $H^4(G_2, \mathbb{Z}) = 0$, implying $\int_{\Sigma_4} \lambda = 0$, for any four-cycle Σ_4 .

Another way to address the problem is to reduce the G_2 solution to a type IIA background. These have a nonvanishing Ramond-Ramond one-form potential from the nontrivial fibration over the M theory circle ∂_γ . One might have hoped to use a similar argument to the MN background by considering the coupling of the one-form potential to D0-branes. However, this will not work because the IIA metric at infinity has vanishing first cohomology $H^1(S^2 \times S^3, \mathbb{Z}) = 0$. This implies that there are no gauge transformations of the one-form potential that are not exact. Therefore none of them will be discretised through quantum mechanical effects. This should be contrasted with the MN situation in which $H^2(S^2 \times S^3, \mathbb{Z}) = \mathbb{Z}$ is the relevant cohomology group.

The conclusion of this subsection is that the $U(1) \rightarrow \mathbb{Z}_{2N}$ breaking appears to be more subtle in the G_2 backgrounds than in, say, the MN solution. Although the effects of the breaking seem to be visible indirectly in the IR, it seems the classical background cannot cause the breaking directly in the UV. An optimistic scenario is that the breaking may be visible after considering M theory corrections to the background *à la* Atiyah-Witten [12]. A pessimistic scenario is that the effects of non-decoupling of gravity are worse for the D6-branes than the non-decoupling of stringy modes for the D5-branes, and therefore that the contamination of the pure gauge theory in the UV results in an explicit rather than anomalous breaking of the $U(1)$ symmetry.

6 Massive isometries: masses from gravitational backgrounds

In this section we develop a theory of massive isometry in a more general context. We show that another example of a massive isometry is found in the Atiyah-Hitchin metric. We give a covariant definition of the mass associated with a massive isometry.

We have argued that spontaneous symmetry breaking along a certain angle in a G_2 holonomy metric is dual to the anomalous chiral $U(1)$ R-symmetry of $\mathcal{N} = 1$ super Yang-Mills theory. One reason for selecting the particular angle is that fluctuations about this, non-isometric,

direction give a reduced Lagrangian describing a massive vector field. Such an effect is known to be generically dual to an anomalous chiral symmetry, although this is the first time that the only background field is the metric, and therefore the angle cannot be an isometry. We call a direction about which one generates a mass term, and about which the fluctuation is consistent to first order, a massive isometry.

We would like massive isometries to be fairly special directions. Otherwise, they do not provide a sharp criterion for identifying appropriate angles. To this end we now study fluctuations about a generic non-isometric direction.

6.1 Fluctuations about a generic direction

Consider the metric

$$ds^2 \equiv G_{uv} dx^u dx^v = h_{mn}(x, \psi) dx^m dx^n + \phi^2(x) (d\psi + K(x) + W(x))^2, \quad (91)$$

where K is part of the ‘background’ metric, and W is the fluctuation. The use of indices is now that $m, n \dots$ to run over all coordinates except ψ . One could also add a perturbation $\lambda(x)$ to ψ , but this may be absorbed into $W(x)$ through the change of coordinates/gauge transformation discussed above in (27). Note the ψ -dependence in h_{mn} which means that this direction is not an isometry. We could have further allowed ϕ and K to depend on ψ , but this will not be necessary for the class of metrics we are interested in.

The metric (91) has vielbeins

$$e^{\bar{m}} = e^{\bar{m}}_n(x, \psi) dx^n \quad e^{\bar{\psi}} = \phi (d\psi + K + W), \quad (92)$$

and inverse vielbeins

$$dx^m = e^m_{\bar{n}} e^{\bar{n}} \quad d\psi = \phi^{-1} e^{\bar{\psi}} - K - W. \quad (93)$$

What is the action describing the fluctuation about a generic non-isometric direction, as in (91)? We will now calculate this action.

In Appendix A we calculate the spin connections for this metric (91) and the Ricci scalar, in order to obtain the action for the fluctuation by substituting into the Einstein-Hilbert action. The result is

$$R(W) = -\frac{1}{4} \phi^2 F_{\bar{s}\bar{m}} F_{\bar{s}\bar{m}} + [2B_{\bar{m}\bar{s}} B_{\bar{n}\bar{s}} - 2B_{\bar{m}\bar{n}} E + (E^2 - B_{\bar{s}\bar{t}} B_{\bar{s}\bar{t}}) \delta_{\bar{m}\bar{n}}] W_{\bar{m}} W_{\bar{n}}. \quad (94)$$

In this expression we have introduced quantities defined in terms of the vielbeins (92). These are

$$E_{\bar{m}\bar{n}} = e^p_{\bar{m}} \partial_{\bar{n}} e^{\bar{n}}_p,$$

$$\begin{aligned}
B_{\bar{m}\bar{n}} &= E_{(\bar{m}\bar{n})}, \\
E &= E_{\bar{m}\bar{m}}.
\end{aligned} \tag{95}$$

The field strength is $F = dW$.

Therefore, we find that mass terms are generic. There are no linear terms because we are perturbing about a solution to Einstein's equations, so the first order change to the action vanishes. However, generating a mass was only the first half of the definition of massive isometry. In order for the perturbation to actually exist classically, it must be consistent, that is, it must solve the full equations of motion. We saw above in studying G_2 metrics, and will see in more generality below, that this condition is indeed rather nontrivial.

6.2 Application to the G_2 cases

As a test of our formula, we may recover the action for the G_2 metrics we discussed previously. To do this, we work with the more specialised metric form

$$ds^2 = dx_4^2 + dr^2 + e^{\bar{m}}(r, y, \psi) e^{\bar{m}}(r, y, \psi) + \phi(y, r)^2 [d\psi + K(r, y) + W(x, r)]^2, \tag{96}$$

where $W(x, r)$ will be the perturbation. We further require that K has no dr component. Consistently with our previous index notation, we let a, b, \dots run over the directions t, x, y, z, r and let m, n, \dots run over the angular directions in the G_2 manifold, which are denoted y in (96). The G_2 metrics have, for example,

$$\phi(y, r) = u(r), \quad K = p(r)d\gamma + q(r)\cos\theta d\phi + s(r)\cos\alpha d\beta, \tag{97}$$

where we just compare the metric (96) with the G_2 metric in the form (22).

The various terms in the action may now be computed

$$\begin{aligned}
E &= 0, \\
B_{\bar{m}\bar{n}} B_{\bar{m}\bar{n}} &= \frac{[b(r)^2 - a(r)^2]^2}{4a(r)^2 b(r)^2}, \\
\phi^2 &= u^2 = c^2 + \frac{(g^2 - 1)^2 [2abf + c(a^2 - b^2)]^2}{4a^2 b^2}.
\end{aligned} \tag{98}$$

Substituting into a suitably specialised version of the expression for the Ricci scalar (94) one obtains

$$\left(\frac{(g^2 - 1)^2 [2abf + c(a^2 - b^2)]^2}{64a^2 b^2} + \frac{c^2}{4} \right) F_{ab} F^{ab} + \frac{[a^2 - b^2]^2}{4a^2 b^2} W_a W^a, \tag{99}$$

We also need the volume factor of the metric, which is

$$(-G)^{1/2} = [g_7(r)]^{1/2} \sin\theta \sin\alpha. \tag{100}$$

Putting these together, and doing the integral over all the angles, we reproduce the Lagrangian found before in (34). Note that the action is Lorentz invariant. This follows from the specialised ansatz (96).

6.3 Application to the Atiyah-Hitchin metric

The Atiyah-Hitchin metric arises in many contexts in string/M theory [45]. It was initially considered as the moduli space space of two $SU(2)$ BPS monopoles in four dimensions [46]. More relevant here will be the fact that when extended with seven flat directions to be a supersymmetric solution of M theory, the background is the strong coupling dual of type IIA string theory on an $O6^-$ plane. It also describes the Coloumb branch of a certain three-dimensional supersymmetric gauge theory.

The metric may be written in the following Bianchi IX form,

$$ds_{11}^2 = dx_7^2 + a(r)^2 b(r)^2 c(r)^2 dr^2 + a(r)^2 \sigma_1^2 + b(r)^2 \sigma_2^2 + c(r)^2 \sigma_3^2. \quad (101)$$

The σ_i are, as previously, left invariant one-forms of $SU(2)$ with Euler angles (θ, ϕ, ψ) . The radial functions satisfy the following first order equations

$$\dot{a} = \frac{a(b^2 + c^2 - a^2)}{2} - abc, \quad + \text{cyclic}. \quad (102)$$

The ranges of the coordinates are $0 \leq \theta \leq \pi$, $0 \leq \phi < 2\pi$ and $0 \leq \psi < 2\pi$, so the symmetry of the manifold is $SO(3)$ rather than $SU(2)$.

As $r \rightarrow \infty$ the metric acquires a further isometry, which is the $U(1)$ generated by ∂_ψ . At finite r , this isometry is broken by an exponentially small value of $a(r)^2 - b(r)^2$. The lack of isometry in the interior of the metric has the important physical consequence of nonconservation of charge of the individual monopoles in two monopole scattering [47]. Of course the total charge is conserved. The corrections should be thought of as tree level exchange of massive gauge bosons. These gauge bosons are the perturbative degrees of freedom of the Higgsed field theory in which the monopoles exist, they have a constant mass and should not be confused with the massive gauge fields to be described shortly. When the monopole scattering is embedded into string theory as intersecting branes, the exponential corrections admit an elegant interpretation in terms of instanton corrections [45].

The behaviour of the direction ∂_ψ is thus very similar to what we found for ∂_ψ in the G_2 holonomy metrics. We will now show that fluctuations about this direction may consistently be described as a massive gauge field. Thus the Atiyah-Hitchin manifold has a massive isometry.

As usual, we work in the $\lambda = 0$ gauge and therefore the fluctuation is just

$$d\psi \rightarrow d\psi + W_a(x, r)dx^a. \quad (103)$$

Substituting into the eleven dimensional Einstein-Hilbert action we find the following action for the fluctuation

$$S \propto - \int d^7x dr \left[\frac{a^2 b^2 c^4}{4} F_{ab} F^{ab} + \frac{(a^2 - b^2)^2 c^2}{2} W_a W^a \right]. \quad (104)$$

The similarity with the $g = 1$ metrics of G_2 holonomy is remarkable. Note that in eight dimensions, indices are raised and lowered with the metric (101) with its nontrivial g^{rr} coefficient.

We now need to check that this reduced action is consistent. This is done similarly to before. We let

$$P(r) = \frac{a^2 b^2 c^4}{4}, \quad Q(r) = \frac{(a^2 - b^2)^2 c^2}{2}. \quad (105)$$

The equations of motion following from the action (104) are then

$$\begin{aligned} \partial_j F^j{}_r &= \frac{Q}{2P} W_r, \\ \partial_r (g^{rr} F_r{}^i) + \partial_j F^{ji} + \frac{\partial_r P}{P} g^{rr} F_r{}^i &= \frac{Q}{2P} W^i. \end{aligned} \quad (106)$$

These imply the divergence equation

$$\partial_i W^i + \partial_r (g^{rr} W_r) + \left[\frac{\partial_r P}{P} + \frac{2P}{Q} \partial_r \left(\frac{Q}{2P} \right) \right] g^{rr} W_r = 0. \quad (107)$$

We may now calculate the fluctuation of the eleven dimensional Ricci tensor. Using the equations of motion we just derived and also the first order equations for the radial functions, we find that

$$\delta R_{\bar{u}\bar{v}} = 0. \quad (108)$$

Therefore the reduced equations imply the full eleven dimensional linearised equations of motion and the fluctuation is consistent.

It is interesting that again the first order radial equations are necessary, showing that supersymmetry of the background is what allows consistency of the fluctuation.

6.4 Consistency in the generic case

In this subsection we examine consistency of a generic massive isometry. It follows from (94) that the generic action for a massive isometry is

$$S \propto - \int dx e\phi \left[\frac{\phi^2}{4} F_{mn} F^{mn} - \frac{n-2}{n} (E^2 - B_{\bar{s}\bar{t}} B_{\bar{s}\bar{t}}) W_m W^m \right], \quad (109)$$

where $e = \det e^{\bar{m}}_k$. Here n denotes the number of m, n, \dots indices. Note that we are now working in curved indices for the gauge fields. We have integrated out the ψ angle, so it is important that the Lagrangian does not depend on this direction, except possibly as an overall factor. The equation of motion following from this action is then

$$\nabla_n F^{nm} + \frac{2(n-2)}{n} \frac{E^2 - B_{\bar{s}\bar{t}} B_{\bar{s}\bar{t}}}{\phi^2} A^m + \left[3 \frac{\partial_n \phi}{\phi} + e^s_{\bar{t}} \partial_n e^{\bar{t}}_s \right] F^{nm} = 0. \quad (110)$$

The equation is perhaps clearer in the case of the restricted ansatz of equation (96). In this case the equations of motion are

$$\partial_a F^{ab} + 2 \frac{E^2 - B_{\bar{m}\bar{n}} B_{\bar{m}\bar{n}}}{\phi^2} A^b + \left[3 \frac{\partial_r \phi}{\phi} + e^s_{\bar{t}} \partial_r e^{\bar{t}}_s \right] F^{rb} = 0, \quad (111)$$

where a, b run over the five directions t, x, y, z, r .

Working in case of (96), we now calculate the linear terms in the fluctuation for the Ricci tensor. These must all vanish for the perturbation to be consistent. Let us look at some of the simpler components

$$\begin{aligned} \delta R_{\bar{\psi}\bar{\psi}} &= \frac{\partial_r \phi}{\phi} E W_{\bar{r}}, \\ \delta R_{\bar{m}\bar{\psi}} &= \frac{1}{2} \phi W_{\bar{r}} (E_{\bar{s}\bar{m}} H_{\bar{s}\bar{r}} + E H_{\bar{r}\bar{m}} + \partial_{\bar{\psi}} H_{\bar{r}\bar{m}}), \\ \delta R_{\bar{r}\bar{r}} &= -(2 B_{\bar{m}\bar{n}} A_{\bar{r}\bar{m}\bar{n}} + \partial_{\bar{\psi}} A_{\bar{r}\bar{s}\bar{s}}) W_r h + \partial_r (E W_{\bar{r}}), \\ \delta R_{\bar{i}\bar{r}} &= \frac{E}{2} (\partial_i W_{\bar{r}} + \partial_r W_{\bar{i}}) - W_{\bar{i}} B_{\bar{m}\bar{n}} A_{\bar{r}\bar{m}\bar{n}}, \\ \delta R_{\bar{i}\bar{\psi}} &= -\frac{\phi}{2} \left[\partial_{\bar{c}} F_{\bar{c}\bar{i}} + F_{\bar{r}\bar{i}} \left(3 \frac{\partial_r \phi}{\phi} + e^s_{\bar{m}} \partial_r e^{\bar{m}}_s \right) + \frac{2(E^2 - B_{\bar{n}\bar{m}} B_{\bar{n}\bar{m}})}{\phi^2} W_{\bar{i}} \right] + \frac{W_{\bar{i}}}{\phi} [\partial_{\bar{\psi}} E + E^2], \\ &= \frac{W_{\bar{i}}}{\phi} [\partial_{\bar{\psi}} E + E^2]. \end{aligned} \quad (112)$$

Note that in the last of these equations we have used the equations of motion for the fluctuation (111). In these expressions we have used the objects

$$\begin{aligned} A_{\bar{m}\bar{n}\bar{s}} &= -e^p_{[\bar{m}} e^q_{\bar{n}]} \partial_p e^{\bar{s}}_q - e^p_{[\bar{m}} e^q_{\bar{s}]} \partial_p e^{\bar{n}}_q + e^p_{[\bar{n}} e^q_{\bar{s}]} \partial_p e^{\bar{m}}_q, \\ H &= dK, \end{aligned} \quad (113)$$

where K is defined in equation (91). We see that in order for these fluctuations to vanish, we need the following independent consistency conditions:

$$E = 0, \quad (114)$$

$$E_{\bar{s}\bar{m}} H_{\bar{s}\bar{r}} + \partial_{\bar{\psi}} H_{\bar{r}\bar{m}} = 0. \quad (115)$$

$$A_{\bar{r}\bar{s}\bar{n}} B_{\bar{s}\bar{n}} = 0, \quad (116)$$

and

$$\partial_\psi A_{\bar{r}\bar{s}\bar{s}} = 0. \quad (117)$$

There are, of course, more consistency conditions which follow from requiring the vanishing of other $\delta R_{\bar{\mu}\bar{\nu}}$ terms. These are in general fairly long and unilluminating. One further simple condition that may be found comes from

$$\frac{W_{\bar{i}}}{W_{\bar{r}}} \delta R_{\bar{r}\bar{n}} - \delta R_{\bar{i}\bar{n}} = \frac{\phi^2}{2} H_{\bar{r}\bar{n}} F_{\bar{r}\bar{i}}, \quad (118)$$

which then implies

$$H_{\bar{r}\bar{n}} = 0. \quad (119)$$

Note that this is sharper than, and implies, equation (115) above.

We may apply these formulae to the G_2 metrics. We find that the first four conditions, (114) to (117), are satisfied by all the G_2 holonomy metrics. The fifth condition (119) is satisfied when the metric function $g(r) = 1$, just as we found previously in section 3.2. In fact, calculating the right hand side of (119) explicitly, one finds precise agreement with (47). This gives a pleasant check on the calculations of this and that section.

It is hopefully apparent from the simpler conditions for consistency we have derived here, and the more complicated conditions we have not discussed, that the requirement that a massive vector fluctuation be consistent, and hence a massive isometry, is nontrivial. The necessary use of the first order supersymmetry equations for the background, see section 3, further supports this statement. Asymptotic consistency, that is, consistency in the $g(r) = 1$ G_2 metrics, is also significantly nontrivial. In particular, there does not seem to be another obvious non-isometric angle in the $g(r) = 1$ metrics that would satisfy the condition (119).

6.5 Towards a covariant description of massive isometry

In previous subsections, we used a particular choice of coordinates in which we perturbed the background solution. The definition we gave above of massive isometry depended on first finding coordinates such that the metric takes on a specific form.

An interesting question is whether a fully covariant description of massive isometry is possible. That is, given a non-isometric direction with tangent vector k , if one takes coordinates adapted to this vector field

$$k = \frac{\partial}{\partial \psi}, \quad (120)$$

is there a collection of covariant conditions that k must satisfy in order for the perturbation $d\psi \rightarrow d\psi + A$ to produce a massive isometry?

Such a description would be analogous to the Killing vector equation for an isometry. If we have a vector field k satisfying $\nabla_u k_v + \nabla_v k_u = 0$, then it is well known that if we take coordinates adapted to this vector field and perturb, then the fluctuation is described by massless electrodynamics.

We will not attempt to find a fully covariant description of massive isometry here. However, we will give a covariant description of the mass of the vector field associated with a massive isometry.

Let k be a massive isometry, consider

$$\nabla_{\bar{u}} k_{\bar{v}} + \nabla_{\bar{v}} k_{\bar{u}} = e^s_{\bar{u}} \partial_s k_{\bar{v}} + e^s_{\bar{v}} \partial_s k_{\bar{u}} + [w_{\bar{u}\bar{v}\bar{s}} + w_{\bar{v}\bar{u}\bar{s}}] k_{\bar{s}}. \quad (121)$$

If we work in the adapted coordinates, assumed to be of the form of (91), then we have $k_{\bar{\psi}} = \phi$ and $k_{\bar{m}} = 0$. Then, using the spin connections of (129) in the Appendix we find that

$$\nabla_{\bar{\psi}} k_{\bar{\psi}} = 0, \quad \nabla_{(\bar{\psi}} k_{\bar{m}}) = 0, \quad \nabla_{(\bar{m}} k_{\bar{n}}) = B_{\bar{m}\bar{n}}, \quad (122)$$

where $B_{\bar{m}\bar{n}}$ is defined in equation (95). Note that this result neatly includes the case when k is an isometry and $B_{\bar{m}\bar{n}}$ is zero. The expression is not tensorial however.

Another expression that may be derived similarly is

$$\nabla_{\bar{u}} k_{\bar{u}} = E = E_{\bar{m}\bar{m}}. \quad (123)$$

Combining these results and referring to equation (94) we see that the mass of the vector field will be proportional to

$$\text{mass}^2 \propto \frac{B_{\bar{m}\bar{n}} B_{\bar{m}\bar{n}} - E^2}{\phi^2} = - \frac{k_{\bar{u}} \nabla_{\bar{v}} \nabla_{\bar{v}} k_{\bar{u}} + k_{\bar{u}} \nabla_{\bar{v}} \nabla_{\bar{u}} k_{\bar{v}} + 2 (\nabla_{\bar{u}}) (k_{\bar{u}} \nabla_{\bar{v}} k_{\bar{v}})}{2 k_{\bar{u}} k_{\bar{u}}}. \quad (124)$$

This result takes on a particularly nice form in the case when $E = 0$, and when the background is Ricci flat. Both these conditions are satisfied for the G_2 metrics. As we have seen, the condition $E = 0$ is in fact necessary for a massive isometry to be consistent. Ricci flatness allows us to use $\nabla_{\bar{u}} \nabla_{\bar{v}} k_{\bar{u}} = \nabla_{\bar{v}} \nabla_{\bar{u}} k_{\bar{u}}$. If these conditions hold then one obtains the following simple expression for the mass

$$\text{mass}^2 \propto \frac{B_{\bar{m}\bar{n}} B_{\bar{m}\bar{n}}}{\phi^2} = - \frac{k \cdot \nabla^2 k}{2 k^2}. \quad (125)$$

7 Conclusions and future directions

The main conclusion of our work is that G_2 holonomy metrics exhibit spontaneous symmetry breaking of a vector fluctuation about a certain non-isometric angle. This is an

important phenomenon within the context of gauge-gravity dualities, because on general grounds anomalous global symmetries in field theory are expected to be dual to spontaneously broken gauge symmetries in gravity. In the G_2 holonomy case, we have argued that the vector fluctuation we considered is dual to the anomalous chiral current of $\mathcal{N} = 1$ super Yang Mills theory. This suggests that G_2 holonomy duality may have a dictionary with similar structure to other better studied dualities, even though the background is not asymptotically Anti-de Sitter. The further elucidation of this dictionary is an interesting question. Can the identification of a vector fluctuation as dual to the chiral current be pushed further to allow the computation of gauge theory two point functions?

We have argued that the appropriate G_2 holonomy metrics to use are the fairly recently constructed \mathbb{D}_7 metrics. The availability of concrete metrics, up to radial functions specified by ODEs, was crucial for our work. These metrics should allow further matchings between M theory on the G_2 manifolds and $\mathcal{N} = 1$ gauge theory. Perhaps one can adapt to the G_2 duality the recent success in calculating the $\mathcal{N} = 1$ beta function from the Maldacena-Núñez background [29, 43]? The concrete metrics should also be useful in the IR of the field theory. So far the topological charges of confining strings and domain walls have been matched. The existence of a metric should allow further elucidation of the quantitative dynamics of the strongly coupled regime.

We found that the fluctuation we studied had a five dimensional interpretation that was consistent with previous results. However, it would be useful to understand the five dimensional perspective better by embedding the fluctuation into a full five dimensional supergravity, on a background that lifted to the G_2 holonomy metrics. It seems plausible that such a supergravity could be constructed by reducing the eleven dimensional theory on $SU(2)$ to get an eight dimensional supergravity, as has already been done to construct G_2 holonomy metrics [36, 38, 39], and then doing a further reduction on another $SU(2)$ to get to five dimensions.

A question that needs further clarification is the breaking of $U(1) \rightarrow \mathbb{Z}_{2N}$. It seems that this is not visible from the classical supergravity solution in the UV. We made some preliminary comments above about whether quantum corrections to the classical G_2 background, based on the work of Atiyah and Witten [12], would induce a C field with the correct structure to cause the $U(1) \rightarrow \mathbb{Z}_{2N}$ breaking. It appears that a systematic generalisation of their work to the new G_2 holonomy metrics is required to settle the issue.

Another set of questions thrown up by this work concerns the notion of massive isometry. This concept is natural when considering duality in purely gravitational backgrounds with

a broken symmetry. Is a covariant description of massive isometry possible? Can one find general conditions for consistency of massive isometries? Does the notion have other applications, perhaps phenomenological? A final issue that would be nice to clarify is the relation between consistency of the fluctuation and supersymmetry of the background that we found above.

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A Calculating the action for a generic fluctuation

This appendix evaluates the Ricci scalar of the metric

$$ds^2 \equiv G_{uv}dx^u dx^v = h_{mn}(x, \psi)dx^m dx^n + \phi^2(x) (d\psi + A(x))^2, \quad (126)$$

that was of interest in section 6.1. Here we have set $A = K + W$, where K is part of the ‘background’ metric, and W is the fluctuation. The indices $m, n \dots$ run over all coordinates except ψ .

The metric (126) has vielbeins

$$e^{\bar{m}} = e^{\bar{m}}_n(x, \psi)dx^n \quad e^{\bar{\psi}} = \phi(d\psi + A), \quad (127)$$

and inverse vielbeins

$$dx^m = e^m_{\bar{n}}e^{\bar{n}} \quad d\psi = \phi^{-1}e^{\bar{\psi}} - A. \quad (128)$$

One may now calculate the spin connections. We find

$$\begin{aligned} \omega_{\bar{m}\bar{n}} &= [A_{\bar{m}\bar{n}\bar{s}} + B_{\bar{s}\bar{n}}A_{\bar{m}} - B_{\bar{s}\bar{m}}A_{\bar{n}} - C_{\bar{m}\bar{n}}A_{\bar{s}}]e^{\bar{s}} + \left[C_{\bar{m}\bar{n}} - \frac{1}{2}\phi^2 F_{\bar{m}\bar{n}}\right]\phi^{-1}e^{\bar{\psi}}, \\ \omega_{\bar{\psi}\bar{m}} &= \left[\frac{1}{2}\phi^2 F_{\bar{m}\bar{n}} - B_{\bar{m}\bar{n}}\right]\phi^{-1}e^{\bar{n}} + D_{\bar{m}}\phi^{-1}e^{\bar{\psi}}. \end{aligned} \quad (129)$$

Where $F = dA$ and we have introduced the following quantities,

$$\begin{aligned}
A_{\bar{m}\bar{n}\bar{s}} &= -e^p_{[\bar{m}}e^q_{\bar{n}]} \partial_p e^{\bar{s}}_q - e^p_{[\bar{m}}e^q_{\bar{s}]} \partial_p e^{\bar{n}}_q + e^p_{[\bar{n}}e^q_{\bar{s}]} \partial_p e^{\bar{m}}_q, \\
E_{\bar{m}\bar{n}} &= e^p_{\bar{m}} \partial_\psi e^{\bar{n}}_p, \\
B_{\bar{m}\bar{n}} &= E_{(\bar{m}\bar{n})}, \quad C_{\bar{m}\bar{n}} = E_{[\bar{m}\bar{n}]}, \\
D_{\bar{m}} &= e^p_{\bar{m}} \partial_p \phi.
\end{aligned} \tag{130}$$

These quantities may be used to calculate a reduced action by substituting into the Einstein-Hilbert action. After neglecting a total derivative, the Einstein-Hilbert action may be written in terms of the spin connection as follows

$$S \propto \int dx d\psi G^{1/2} [\omega^{\bar{u}}_{\bar{v}\bar{s}} \omega^{\bar{v}\bar{s}}_{\bar{u}} + \omega^{\bar{u}}_{\bar{u}\bar{v}} \omega^{\bar{s}}_{\bar{s}}^{\bar{v}}]. \tag{131}$$

Where

$$\omega_{\bar{v}\bar{s}} = \omega^{\bar{u}}_{\bar{v}\bar{s}} e^{\bar{u}}. \tag{132}$$

For our metric ansatz (126) we have $G^{1/2} = h^{1/2} \phi$ and the spin connection terms give

$$\omega_{\bar{s}\bar{m}\bar{n}} \omega_{\bar{m}\bar{n}\bar{s}} + \omega_{\bar{m}\bar{m}\bar{s}} \omega_{\bar{n}\bar{n}\bar{s}} + 2\omega_{\bar{m}\bar{\psi}\bar{s}} \omega_{\bar{\psi}\bar{s}\bar{m}} + \omega_{\bar{m}\bar{s}\bar{\psi}} \omega_{\bar{\psi}\bar{s}\bar{m}} + 2\omega_{\bar{m}\bar{m}\bar{s}} \omega_{\bar{\psi}\bar{\psi}\bar{s}} + \omega_{\bar{m}\bar{m}\bar{\psi}} \omega_{\bar{s}\bar{s}\bar{\psi}}, \tag{133}$$

where we have lowered all the flat indices using the flat metric. This expression is found to be

$$\begin{aligned}
&\phi^{-2} [E^2 - B_{\bar{m}\bar{s}} B_{\bar{m}\bar{s}}] + 2\phi^{-1} D_{\bar{s}} A_{\bar{m}\bar{s}\bar{m}} + A_{\bar{m}\bar{s}\bar{m}} A_{\bar{n}\bar{s}\bar{n}} + A_{\bar{m}\bar{n}\bar{s}} A_{\bar{n}\bar{s}\bar{m}} \\
&\quad - \frac{1}{4} \phi^2 F_{\bar{s}\bar{m}} F_{\bar{s}\bar{m}} + F_{\bar{s}\bar{m}} C_{\bar{s}\bar{m}} \\
&\quad + [2(\phi^{-1} D_{\bar{s}} + A_{\bar{m}\bar{s}\bar{m}})(E_{\bar{s}\bar{n}} - E \delta_{\bar{s}\bar{n}}) + 2A_{\bar{m}\bar{n}\bar{s}} E_{\bar{m}\bar{s}}] A_{\bar{n}} \\
&\quad + [2B_{\bar{m}\bar{s}} B_{\bar{n}\bar{s}} - 2B_{\bar{m}\bar{n}} E + (E^2 - B_{\bar{s}\bar{t}} B_{\bar{s}\bar{t}}) \delta_{\bar{m}\bar{n}}] A_{\bar{m}} A_{\bar{n}}.
\end{aligned} \tag{134}$$

We have introduced $E = E_{\bar{m}\bar{m}}$. Note that the expression contains terms with no A dependence, with linear and with quadratic A dependence, and with linear and quadratic F dependence. Recall that $A = K + W$, where K is part of the background and W is the fluctuation. Because we are perturbing about a solution, we know that the linear terms in a perturbation of the action must vanish. Further, the zeroth-order terms may be collected into the background Ricci scalar. Thus the result is in fact the simple expression

$$R(w) = R(W=0) - \frac{1}{4} \phi^2 F_{\bar{s}\bar{m}} F_{\bar{s}\bar{m}} + [2B_{\bar{m}\bar{s}} B_{\bar{n}\bar{s}} - 2B_{\bar{m}\bar{n}} E + (E^2 - B_{\bar{s}\bar{t}} B_{\bar{s}\bar{t}}) \delta_{\bar{m}\bar{n}}] W_{\bar{m}} W_{\bar{n}}, \tag{135}$$

where now we use $F = dW$. This is the result we quoted in (94).

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